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Preface

Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M. Let V be a smooth vector bundle over M. Choose a system of local coordinates $x = (x_1, ..., x_m)$ for M and a local trivialization of V. We say that a second order partial differential operator D on $C^\infty(V)$ is of *Laplace type* if locally we can express

$$D = -\left(\sum_{\mu\nu} g^{\mu\nu} \cdot \mathrm{Id} \frac{\partial^2}{\partial x_\mu \partial x_\nu} + \sum_\mu A^\mu \frac{\partial}{\partial x_\mu} + B\right) .$$

Here A^μ and B are matrix valued. The condition that D is of Laplace type is invariantly defined; it neither depends on the particular coordinate system which was chosen nor on the particular trivialization of V which was chosen. Equivalently, D is of Laplace type if the *leading symbol* of D is scalar and given by the metric tensor.

Let $D_{\mathcal{B}}$ be the realization of an operator D of Laplace type with respect to a suitable boundary condition. Let $\phi \in C^\infty(V)$ describe the initial temperature distribution. Since $D_{\mathcal{B}}$ is strongly elliptic, the subsequent temperature distribution $u := e^{-tD_{\mathcal{B}}}\phi$ for $t \geq 0$ is characterized by the equations:

$$(\partial_t + D)u = 0,\ u|_{t=0} = \phi,\ \text{and } \mathcal{B}u = 0\,.$$

Let $\rho \in C^\infty(V^*)$ be the specific heat of the manifold and let dx be the Riemannian measure on M. The total *heat content*

$$\beta(t) := \int_M e^{-tD_{\mathcal{B}}}\langle\phi, \rho\rangle dx$$

has a complete asymptotic expansion as $t \downarrow 0$ of the form:

$$\beta(t) \sim \sum_{n=0}^\infty \beta_n(\phi, \rho, D, \mathcal{B}) t^{n/2}.$$

The operator $e^{-tD_{\mathcal{B}}}$ is of trace class on $L^2(V)$. Let F be a smooth *smearing endomorphism* which is used for localization and for recovering divergence terms. As $t \downarrow 0$ there is a complete asymptotic expansion of the form:

$$\mathrm{Tr}_{L^2}(Fe^{-tD_{\mathcal{B}}}) \sim \sum_{n=0}^\infty a_n(F, D, \mathcal{B}) t^{(n-m)/2}.$$

Both the *heat content coefficients* β_n and the *heat trace coefficients* a_n are locally computable. The determination of these coefficients for various boundary conditions will comprise the remainder of this book. In Chapter 1, we outline the analytic and differential geometric preliminaries we shall need. In Chapter 2, we study the heat content coefficients β_n, and in Chapter 3, we study the heat trace asymptotics a_n.

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Asymptotic Formulae in Spectral Geometry

Peter B Gilkey

Contents

Chapter 1

Analytic preliminaries

1.0 Introduction

In Chapter 1, we present the necessary differential geometric and analytic facts which we shall need in Chapter 2 and in Chapter 3. Throughout this book, we shall let M be a smooth compact m dimensional Riemannian manifold with smooth boundary ∂M; we say M is a *closed manifold* if ∂M is empty. We adopt the *Einstein convention* and sum over repeated indices.

In Section 1.1 we introduce some of the notions from differential geometry that we shall need. We discuss the Levi-Civita connection, the Riemann curvature tensor, geodesic normal coordinates, and the second fundamental form. We study the interior and the boundary geometries. We introduce Clifford module structures and show that there exist compatible connections.

Let $x = (x_1, ..., x_m)$ be a system of local coordinates on M. Let $\partial_\nu^x := \frac{\partial}{\partial x_\nu}$ and $g_{\mu\nu} := g(\partial_\mu^x, \partial_\nu^x)$. Let $g^{\mu\nu}$ be the inverse matrix; if δ_σ^μ is the *Kronecker symbol*, then $g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu$. Let dx be the Riemannian measure.

Let V be a smooth vector bundle over M. We denote the space of smooth sections to V by $C^\infty(V)$. By an abuse of notation, we will let $C^\infty(M)$ denote the space of smooth functions on M. Let $C_0^\infty(V)$ denote the space of smooth sections to V with compact support in the interior of M. Choose a local trivialization of V. We say that a second order partial differential operator D on $C^\infty(V)$ is of *Laplace type* if locally D has the form

$$D = -(g^{\mu\nu}\mathrm{Id}\,\partial_\mu^x \partial_\nu^x + a^\mu \partial_\mu^x + b)\,.$$

A first order operator A is said to be of *Dirac type* if A^2 is of Laplace type. We may expand any first order operator in the form

$$A = \gamma^\nu \partial_\nu^x + \phi\,.$$

The operator A is of Dirac type if and only if $\gamma := \{\gamma^\nu\}$ gives V a *Clifford* *module structure*; this means we have the commutation relations

$$\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = -2g^{\mu\nu}\mathrm{Id}_V\,.$$

In Section 1.2, we shall study the geometry of operators of Laplace type and the geometry of operators of Dirac type. We show that an operator of Laplace type can be completely described by the data (g, ∇, E) where the Riemannian metric g describes the leading symbol, the connection ∇ on the auxiliary vector bundle V corrects the first order terms, and the endomorphism E of V is the invariantly described 0^{th} order part of the operator. We then introduce the dual structures defining the dual operator on the dual bundle. We express the form valued Laplacian in terms of this formalism; this recovers the Weitzenböck formula. We conclude Section 1.2 with a brief discussion of some of the singular structures which appear in transmission and transfer boundary conditions.

Section 1.3 deals with questions of interior ellipticity. Let M be a closed Riemannian manifold. We define the symbolic spectrum $\mathrm{Spec}_\sigma(P)$ of a partial differential operator P; P is said to be elliptic if $0 \notin \mathrm{Spec}_\sigma(P)$. Note that operators of Laplace type and of Dirac type are elliptic. We discuss elliptic regularity and describe the discrete spectral resolution of a self-adjoint elliptic operator. If the symbolic spectrum of P is $(0, \infty)$, we define the fundamental solution of the heat equation e^{-tP}. We introduce the heat trace asymptotics in this context and relate these invariants to index theory.

Applying these results to the form valued Laplacian then yields a local formula for the Euler-Poincaré characteristic of the manifold that we will use subsequently in Section 1.9 to give a heat equation proof of the Chern-Gauss-Bonnet theorem. We conclude Section 1.3 by discussing the heat content asymptotics.

If the boundary of M is not empty, we must introduce suitable boundary conditions. In Section 1.4, we define the generalized Lopatinskij-Shapiro condition we shall use to discuss the notion of ellipticity with respect to a suitable complex cone. We then focus the discussion on the context of first order operators of Dirac type and on second order operators of Laplace type. We discuss the dual boundary condition for the dual boundary operator and the Green's formula. We extend the heat trace and the heat content asymptotics to the context of elliptic boundary value problems.

In Section 1.5 we introduce some of the various boundary conditions we shall be considering in this book in more detail. We begin by discussing the classical boundary conditions; these are Dirichlet, Neumann, Robin, and mixed boundary conditions. Note that absolute and relative boundary conditions, which appear in the study of the de Rham complex, are special cases of mixed boundary conditions. In Section 1.6, we discuss bag boundary conditions and spectral boundary conditions; these are boundary conditions which arise in the study of operators of Dirac type. Transmission and transfer boundary conditions are introduced. These boundary conditions arise in the study of certain heat conduction problems. Oblique boundary conditions are also discussed. Each of these boundary conditions is shown to satisfy the ellipticity conditions discussed in Section 1.4.

Section 1.7 gives a brief introduction to invariance theory. We describe the first and second main theorems of invariance theory due to H. Weyl. We use dimensional analysis and the first main theorem of invariance theory to discuss various spaces of local invariants of a Riemannian manifold which arise in Chapter 2 and in Chapter 3 in the study of the heat content and the heat trace asymptotics.

In Section 1.8, we begin our study of the heat supertrace asymptotics of the Witten Laplacian by using the second main theorem of invariance theory to obtain spanning sets for certain spaces of invariants which arise in this context. We shall conclude our study of these invariants subsequently in Section 3.8. We conclude Chapter 1 with Section 1.9 where we use the results established previously to give a heat equation proof of the Chern-Gauss-Bonnet theorem for manifolds with boundary.

1.1 Riemannian geometry

1.1.1 The interior geometry

Let M be a compact smooth Riemannian manifold of dimension m with smooth boundary ∂M. We adopt the following *indexing conventions*. Let $x = (x_1, ..., x_m)$ be a system of local coordinates on M. We let Greek indices μ, ν, etc. range from 1 through m and index the coordinate frames ∂_ν^x and dx^ν for the tangent bundle TM and cotangent bundle T^*M, respectively. Let $g_{\mu\nu} := g(\partial_\mu^x, \partial_\nu^x)$. We use the metric to identify the tangent and cotangent bundles and thereby extend the metric to T^*M. Let $g^{\nu\mu}$ be the inverse matrix and let $\xi = \xi_\nu dx^\nu$ be a cotangent vector. Then

$$|\xi|^2 = g^{\nu\mu}\xi_\nu\xi_\mu\,.$$

The Riemannian measure on M is given by

$$dx := \sqrt{\det g_{\mu\nu}}dx^1...dx^m\,.$$

If M has non-empty boundary ∂M, the Riemannian measure dy of ∂M is defined similarly.

Let ∇ be the *Levi-Civita connection* of M. We shall also use the notation ∇^M or ∇^g occasionally when it is necessary to specify the manifold or geometry under consideration. Let Γ denote the associated *Christoffel symbols* of the first and of the second kind

$$\begin{aligned}
&g(\nabla_{\partial_\nu^x}\partial_\mu^x, \partial_\sigma^x) = \Gamma_{\nu\mu\sigma} \text{ where } \Gamma_{\nu\mu\sigma} = \tfrac{1}{2}(\partial_\nu^x g_{\mu\sigma} + \partial_\mu^x g_{\nu\sigma} - \partial_\sigma^x g_{\nu\mu}),\\
&\nabla_{\partial_\nu^x}\partial_\mu^x = \Gamma_{\nu\mu}{}^\sigma\partial_\sigma^x \quad \text{where} \quad \Gamma_{\nu\mu}{}^\sigma = g^{\sigma\varepsilon}\Gamma_{\nu\mu\varepsilon}, \quad \text{and} \qquad (1.1.a)\\
&\nabla_{\partial_\nu^x}dx^\mu = \Gamma_\nu{}^\mu{}_\sigma dx^\sigma \quad \text{where} \quad \Gamma_\nu{}^\mu{}_\sigma = -\Gamma_{\nu\sigma}{}^\mu\,.
\end{aligned}$$

Let R be the *curvature* of the Levi-Civita connection,

$$R(x,y,z,w) = g((\nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]})z,w)\,.$$

One has the following *curvature symmetries*:

$$\begin{aligned} &R(x,y,z,w) = R(z,w,x,y) = -R(y,x,z,w),\\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0\,. \end{aligned} \tag{1.1.b}$$

The first symmetries are $\mathbb{Z}_2$ symmetries; the final symmetry is called the *first Bianchi identity.*

A 4 tensor satisfying the symmetries of Display (1.1.b) is said to be an *algebraic curvature tensor.* Every algebraic curvature tensor is geometrically realizable [191] in the following sense. Suppose given a positive definite inner product g_V and an algebraic curvature tensor R_V on an m dimensional real vector space V. Then there exists the germ of a Riemannian metric g on $\mathbb{R}^m$ and an isometry ϕ from $T_0(\mathbb{R}^m)$ to V so that $\phi^*g_V = g$ and $\phi^*R_V = R$. Consequently, the symmetries of Display (1.1.b) generate all the universal symmetries of the curvature tensor.

We define the *Ricci tensor* ρ and the *scalar curvature* τ by setting

$$\rho_{\mu\nu} := g^{\sigma\varepsilon}R_{\mu\sigma\varepsilon\nu} \quad \text{and} \quad \tau := g^{\mu\nu}\rho_{\mu\nu}\,.$$

The components of R, relative to the coordinate frame ∂_μ^x, are given by

$$\begin{aligned} R_{\mu\nu\sigma\varepsilon} : \;&=\; g((\nabla_{\partial_\mu^x}\nabla_{\partial_\nu^x} - \nabla_{\partial_\nu^x}\nabla_{\partial_\mu^x})\partial_\sigma^x, \partial_\varepsilon^x)\\ &=\; g_{\varrho\varepsilon}\{\partial_\mu^x\Gamma_{\nu\sigma}{}^\varrho - \partial_\nu^x\Gamma_{\mu\sigma}{}^\varrho + \Gamma_{\mu\theta}{}^\varrho\Gamma_{\nu\sigma}{}^\theta - \Gamma_{\nu\theta}{}^\varrho\Gamma_{\mu\sigma}{}^\theta\}. \end{aligned} \tag{1.1.c}$$

Similarly, we may define the *covariant derivative curvature tensor* ∇R by

$$\begin{aligned} \nabla R(x,y,z,w;v) : \;&=\; vR(x,y,z,w) - R(\nabla_v x,y,z,w)\\ &-\; R(x,\nabla_v y,z,w) - R(x,y,\nabla_v z,w)\\ &-\; R(x,y,z,\nabla_v w)\,. \end{aligned}$$

This tensor has the symmetries

$$\begin{aligned} &\nabla R(x,y,z,w;v) = \nabla R(z,w,x,y;v) = -\nabla R(y,x,z,w;v),\\ &\nabla R(x,y,z,w;v) + \nabla R(y,z,x,w;v) + \nabla R(z,x,y,w;v) = 0,\\ &\nabla R(x,y,z,w;v) + \nabla R(x,y,w,v;z) + \nabla R(x,y,v,z;w) = 0\,. \end{aligned} \tag{1.1.d}$$

The first symmetries of Display (1.1.d) arise from covariantly differentiating the symmetries of Display (1.1.b). The final symmetry is a new symmetry called the *second Bianchi identity.* A 5 tensor satisfying these symmetries is said to be an *algebraic covariant derivative curvature tensor.* Again, as every such tensor is geometrically realizable [191], the symmetries of Display (1.1.d) generate all the universal symmetries of the covariant derivative of the curvature tensor. We can proceed in this fashion to define the higher covariant derivative curvature tensors $\nabla^\nu R$ for any ν and to derive the appropriate universal symmetries.

We use the metric to identify the tangent and cotangent bundles. Let Roman indices i, j, etc. range from 1 to m and index an orthonormal frame $\{e_i\}$ for these bundles. *Geodesic coordinates* $x = (x_1, ..., x_m)$ centered at a point x_0 are characterized by the property that the rays $\gamma(t) = (tx_1, ..., tx_m)$ are geodesics from x_0 with $\dot\gamma(0) = x_1 e_1 + ... + x_m e_m$. For such a coordinate system we have, see for example [191] Lemma 1.11.4, that

$$\begin{aligned}
&g_{\mu\nu}(x_0) = \delta_{\mu\nu},\\
&(\partial_\sigma^x g_{\mu\nu})(x_0) = 0,\\
&(\partial_\nu^x \partial_\varepsilon^x g_{\mu\sigma})(x_0) = \tfrac{1}{3}\{R_{\mu\nu\sigma\varepsilon} - R_{\mu\varepsilon\nu\sigma}\}(x_0),\\
&(\partial_\nu^x \partial_\varepsilon^x \partial_\varrho^x g_{\mu\sigma})(x_0) = \tfrac{1}{3}\{R_{\mu\nu\sigma\varepsilon;\varrho} - R_{\mu\varepsilon\nu\sigma;\varrho}\}(x_0),\\
&R_{\mu\nu\sigma\varepsilon}(x_0) = \{\partial_\nu^x \partial_\varepsilon^x g_{\mu\sigma} - \partial_\nu^x \partial_\sigma^x g_{\mu\varepsilon}\}(x_0),\\
&R_{\mu\nu\sigma\varepsilon;\varrho}(x_0) = \{\partial_\nu^x \partial_\varepsilon^x \partial_\varrho^x g_{\mu\sigma} - \partial_\nu^x \partial_\sigma^x \partial_\varrho^x g_{\mu\varepsilon}\}(x_0)\,.
\end{aligned}$$

We refer to Atiyah, Bott, and Patodi [7] for the proof of the following result which shows that all the jets of the metric are expressible in terms of the covariant derivatives of the curvature tensor in geodesic coordinates:

Theorem 1.1.1 *Let x be a system of geodesic coordinates centered at a point x_0 of a Riemannian manifold. Then $g_{\mu\nu}(x_0) = \delta_{\mu\nu}$ and $\partial_\sigma^x g_{\mu\nu}(x_0) = 0$. If $\ell \geq 2$, then $(\partial_{\sigma_1}^x ... \partial_{\sigma_\ell}^x g_{\mu\nu})(x_0)$ is expressible in terms of the components of R and of the covariant derivatives of R at x_0.*

1.1.2 The boundary geometry

If $y \in \partial M$, then let $\gamma_y(t)$ be the geodesic starting at y so that $\dot\gamma_y(0)$ is the inward unit normal. Since M is compact, there exists a uniform $\epsilon > 0$ so that $\gamma_y(t)$ is defined for $t \in [0, \epsilon)$ for all $y \in \partial M$. The map $\partial M \times [0, \epsilon) \to \gamma_y(t)$ then defines a diffeomorphism between $\partial M \times [0, \epsilon)$ and a neighborhood of ∂M in M called the *collared neighborhood.*

Let $y = (y_1, ..., y_{m-1})$ be a system of local coordinates on ∂M. The collaring induces a system of coordinates $x = (y_1, ..., y_{m-1}, x_m)$ on M which are called a *normalized coordinate system* where x_m is the geodesic distance to the boundary. These coordinates are characterized by the property that the curves $x_m \to (y, x_m)$ are unit speed geodesics which are perpendicular to the boundary. We adopt the *indexing convention* that Greek indices α, β, etc. range from 1 through $m-1$ and index the associated coordinate frames $\{\partial_\alpha^x\}$ and $\{dx^\alpha\}$ for the tangent bundles and cotangent bundles of the boundary.

Near the boundary, we normalize the orthonormal frame for TM so that $e_m = \partial_m^x$ is the inward *unit geodesic normal vector field.* We shall let Roman indices a, b, etc. range from 1 through $m-1$ and index the induced orthonormal frame $\{e_1, ..., e_{m-1}\}$ for the tangent and cotangent bundles of the boundary.

The *second fundamental form* L is the tensor field whose components relative to a normalized coordinate frame are given by

$$L_{\alpha\beta} := g(\nabla_{\partial_\alpha^x} \partial_\beta^x, \partial_m^x) = \Gamma_{\alpha\beta m}\,. \tag{1.1.e}$$

For example, if D^2 is the unit disk in $\mathbb{R}^2$, then we may introduce polar coordinates $T(r,\theta) := (r\cos\theta, r\sin\theta)$; $-\partial_r$ is the inward geodesic normal on $\partial M = S^1$. Since $g(\partial_\theta,\partial_\theta) = r^2$, the *second fundamental form of the disk* is

$$L(\partial_\theta,\partial_\theta)|_{r=1} = \tfrac{1}{2}\partial_r(r^2)|_{r=1} = +1.$$

More generally, the second fundamental form of the boundary of the ball of radius ρ in $\mathbb{R}^m$ is given by $\rho^{-1}\delta_{ab}$.

The normalized *mean curvature* κ can be defined either with relation to an orthonormal frame or with respect to a coordinate frame by setting

$$\kappa := L_{aa} = g^{\alpha\beta}L_{\alpha\beta}\,.$$

We refer to Section 1.7 for further details concerning the construction of scalar invariants by contracting indices.

Lemma 1.1.2 *Let $y = (y_1, ..., y_{m-1})$ be local coordinates on ∂M. Let x_m be the geodesic distance to the boundary and let $x = (y_1, ..., y_{m-1}, x_m)$ give local coordinates on M. Then*

1. $ds_M^2 = g_{\alpha\beta}(x)dy^\alpha \circ dy^\beta + dx^m \circ dx^m$.

2. $L_{\alpha\beta} = -\frac{1}{2}\partial_m^x g(\partial_\alpha,\partial_\beta)$.

3. Let $g := \sqrt{\det g_{\alpha\beta}}$. Then $\kappa = -\partial_m^x \ln g$.

Proof: Since the curves $t \to (y,t)$ are unit speed geodesics which are perpendicular to the boundary,

$$\nabla_{\partial_m^x}\partial_m^x = 0, \quad g_{\alpha m}(y,0) = 0, \quad \text{and} \quad g_{mm}(y,0) = 1\,.$$

We may therefore compute

$$\partial_m^x g(\partial_m^x,\partial_m^x) = 2g(\nabla_{\partial_m^x}\partial_m^x,\partial_m^x) = 0\,.$$

Since $g(\partial_m^x,\partial_m^x) = 1$ when $x_m = 0$, $g(\partial_m^x,\partial_m^x) \equiv 1$. We may also compute

$$\begin{aligned}\partial_m^x g(\partial_m^x,\partial_\alpha^x) &= g(\nabla_{\partial_m^x}\partial_m^x,\partial_\alpha^x) + g(\partial_m^x,\nabla_{\partial_m^x}\partial_\alpha^x)\\ &= 0 + g(\partial_m^x,\nabla_{\partial_\alpha^x}\partial_m^x) = \tfrac{1}{2}\partial_\alpha^x g(\partial_m^x,\partial_m^x)\\ &= 0\,.\end{aligned}$$

Since $g(\partial_m^x,\partial_\alpha^x) = 0$ when $x_m = 0$, we see $g(\partial_m^x,\partial_\alpha^x) \equiv 0$. This establishes Assertion (1).

We may derive Assertion (2) from Assertion (1) by computing

$$\begin{aligned}L_{\alpha\beta} &= g(\nabla_{\partial_\alpha^x}\partial_\beta^x,\partial_m^x) = \Gamma_{\alpha\beta m}\\ &= \tfrac{1}{2}\{\partial_\alpha^x g(\partial_\beta^x,\partial_m^x) + \partial_\beta^x g(\partial_\alpha^x,\partial_m^x) - \partial_m^x g(\partial_\alpha^x,\partial_\beta^x)\}\\ &= -\tfrac{1}{2}\partial_m^x g(\partial_\alpha^x,\partial_\beta^x)\,.\end{aligned}$$

Making a constant linear change of coordinates in the y variables affects neither κ nor $\partial_m \ln g$. Consequently, in proving Assertion (3), we may assume $g_{\alpha\beta}(y_0,0) = \delta_{\alpha\beta}$ at the point $y_0 \in \partial M$ in question. We compute

$$\begin{aligned}&g_{\alpha\beta}(y_0,x_m) = \delta_{\alpha\beta} + \partial_m g_{\alpha\beta}(y_0)x_m + O(x_m^2),\\ &\det(g_{\alpha\beta})(y_0,x_m) = 1 + \partial_m g_{\alpha\alpha}(y_0)x_m + O(x_m^2),\end{aligned}$$

$$g(y_0,x_m) := \sqrt{\det(g_{\alpha\beta})}(y_0,x_m) = 1 + \tfrac{1}{2}\partial_m g_{\alpha\alpha}(y_0)x_m + O(x_m^2),$$
$$\{\partial_m^x \ln g\}(y_0,x_m) = \{g^{-1}\partial_m^x g\}(y_0,x_m) = \tfrac{1}{2}\partial_m g_{\alpha\alpha}(y_0) + O(x_m),$$
$$\kappa(y_0) = -\tfrac{1}{2}\{g^{\alpha\beta}\partial_m^x g_{\alpha\beta}\}(y_0,0) = -\tfrac{1}{2}\partial_m^x g_{\alpha\alpha}(y_0)\,.$$

Assertion (3) now follows as the point y_0 of the boundary was arbitrary. □

Let ∇_M and $\nabla_{\partial M}$ be the Levi-Civita connections of the metrics on M and on ∂M, respectively. Let R_M and $R_{\partial M}$ be the associated curvature tensors. Let L be the second fundamental form. We may express $R_{\partial M}$ in terms of R_M and L; we refer to Lemma 1.1.4 below. We use ∇_M to covariantly differentiate R_M and we use $\nabla_{\partial M}$ to covariantly differentiate L tangentially. Let $ds^2_{\partial M}$ be the restriction of the metric on M to ∂M. Theorem 1.1.1 generalizes to this setting to become:

Theorem 1.1.3 *Let y be a system of geodesic coordinates on ∂M for the metric $ds^2_{\partial M}$ which are centered at a point $y_0 \in \partial M$. Let $x = (y, x_m)$ where x_m is the geodesic distance to the boundary. Then*

$$g_{\alpha m} \equiv 0, \quad g_{mm} \equiv 1, \quad g_{\alpha\beta}(y_0) = \delta_{\alpha\beta},$$
$$\partial_\gamma g_{\alpha\beta}(y_0) = 0, \quad \textit{and} \quad \partial_m g_{\alpha\beta}(y_0) = -2L_{\alpha\beta}\,.$$

If $\ell \geq 2$, then $(\partial_{\sigma_1}^x ... \partial_{\sigma_\ell}^x)(g_{\mu\nu})(y_0)$ is expressible in terms of the components of the following tensors at y_0

$$\{R_M, \nabla_M R_M, \nabla_M^2 R_M, ..., L, \nabla_{\partial M} L, \nabla_{\partial M}^2 L, ...\}\,.$$

1.1.3 Covariant Differentiation

We will often have an auxiliary connection which is given on the auxiliary vector bundle V. We use this connection and the Levi-Civita connection to covariantly differentiate tensor fields ϕ of all types; we shall denote the components of multiple covariant differentiation relative to a local frame $\{e_1, ..., e_m\}$ by $\phi_{;i_1 i_2 ...}$. Thus, for example, if ϕ is a section to V, then $\nabla\phi$ is a section to $T^*M \otimes V$. Consequently,

$$\phi_{;ij} = \nabla_{e_j}\nabla_{e_i}\phi - \nabla_{\{\nabla_{e_j} e_i\}}\phi\,. \tag{1.1.f}$$

We can *commute covariant derivatives* by introducing additional terms involving curvature. Suppose that f is a scalar function. Then ∇f is a section to T^*M. Thus Equation (1.1.f) implies that

$$f_{;ijk} - f_{;ikj} = R_{jkil} f_{;l}\,. \tag{1.1.g}$$

Similarly we may derive the identity:

$$\begin{aligned} R_{i_1 i_2 i_3 i_4; j_1 j_2} &= R_{i_1 i_2 i_3 i_4; j_2 j_1} + R_{j_1 j_2 i_1 k} R_{k i_2 i_3 i_4} \\ &+ R_{j_1 j_2 i_2 k} R_{i_1 k i_3 i_4} + R_{j_1 j_2 i_3 k} R_{i_1 i_2 k i_4} \\ &+ R_{j_1 j_2 i_4 k} R_{i_1 i_2 i_3 k}\,. \end{aligned} \tag{1.1.h}$$

If ϕ is a tensor field which is defined only on the boundary, then we shall use the connection on V and the Levi-Civita connection of the boundary to tangentially covariantly differentiate ϕ and denote the components of multiple tangential differentiation by $\phi_{:a_1a_2\dots}$. We shall omit the proof of the following Lemma in the interest of brevity as it is well known.

Lemma 1.1.4

1. $R_{M,abcm} = L_{bc:a} - L_{ac:b}$.

2. $(\nabla^M_{\partial^x_\alpha} - \nabla^{\partial M}_{\partial^x_\alpha})\partial^x_\beta = L_{\alpha\beta}\partial^x_m$.

3. If f is a scalar function, then $f_{;ab} = f_{:ab} - L_{ab}f_{;m}$.

4. We have $R_{M,abcd} = R_{\partial M,abcd} - L_{ad}L_{bc} - L_{ac}L_{bd}$.

Let r be the rank of the vector bundle V. Let $\vec{s} = (s_1, ..., s_r)$ be a local frame for V. If $A \in \text{End}\,(V)$, then we may expand

$$As_u = A_{uv}s_v\,.$$

The matrix $A^{\vec{s}} := (A_{uv})$ expresses the action of A relative to the local frame $\vec{s}$. If ∇ is a connection on V, then we may expand

$$\nabla_{e_i}s_u = \omega_{i,uv}s_v\,.$$

Let $\omega_i^{\vec{s}} := (\omega_{i,uv})$ be the *connection 1 form* of ∇ relative to $\vec{s}$

$$\omega^{\vec{s}} := e_i \otimes \omega_i^{\vec{s}} \in T^*M \otimes \text{End}\,(V)\,.$$

This depends on the local frame; $\omega^{\vec{s}}$ is not tensorial.

1.1.4 Clifford Algebras

Let V be a finite dimensional real vector space which is equipped with a positive definite inner product $(\cdot,\cdot)$. The *Clifford Algebra*, $\text{Clif}\,(V)$, is the universal unital real algebra generated by V subject to the *Clifford commutation relations*

$$v_1 * v_2 + v_2 * v_1 = -2(v_1, v_2)\cdot \text{Id}\,.$$

A *Clifford module* for V is an auxiliary real, complex, or quaternionic vector space W together with a linear map γ from V to $\text{Hom}\,(W)$ so that the Clifford commutation relations are preserved. This means that

$$\gamma(v_1)\gamma(v_2) + \gamma(v_2)\gamma(v_1) = -2(v_1, v_2)\cdot \text{Id}_W \quad \text{for all} \quad v_1, v_2 \in V\,.$$

The map γ then extends to a unital algebra homomorphism from $\text{Clif}\,(V)$ to $\text{Hom}\,(W)$. If $\{e_1, ..., e_m\}$ is an orthonormal basis for V, then γ is specified by giving elements $\gamma_i := \gamma(e_i) \in \text{Hom}\,(W)$ so that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\cdot \text{Id}_W\,.$$

We can now determine the *complex Clifford modules*. We first suppose $\dim V$ to be even.

Lemma 1.1.5 *Let $m := \dim(V) = 2\bar{m}$ be even. There exists a unique irreducible complex Clifford module $\mathcal{S}$ for V with* $\dim \mathcal{S} = 2^{\bar{m}}$ *so that if W is any complex Clifford module for V, then $W = \ell \cdot \mathcal{S}$.*

Proof: We follow the discussion in Atiyah, Bott, and Shapiro [8]; we shall use a similar argument in our discussion of bag boundary conditions in Section 1.6.5 presently. Let $\{e_1, ..., e_m\}$ be an orthonormal basis for V. Let γ give W a Clif (V) module structure. Set $\gamma_j := \gamma(e_j)$. We define

$$\tau_i := \sqrt{-1}\gamma_{2i-1}\gamma_{2i} \in \operatorname{End}(W)\,.$$

The elements $\{\tau_1, ..., \tau_{\bar{m}}\}$ are a commuting family of linear endomorphisms of W satisfying $\tau_i^2 = \operatorname{Id}_W$. Let $\vec{\varrho} = (\varrho_1, ..., \varrho_{\bar{m}})$ where $\varrho_i = \pm 1$; there are $2^{\bar{m}}$ possible such collections of signs. We decompose

$$W = \oplus_{\vec{\varrho}} W_{\vec{\varrho}} \quad \text{where} \quad W_{\vec{\varrho}} := \{w \in W : \tau_i w = \varrho_i w\}$$

are the simultaneous eigenspaces. We have

$$\gamma_j \tau_i = \begin{cases} -\tau_i \gamma_j & \text{if } j = 2i-1, 2i, \\ \tau_i \gamma_j & \text{otherwise}\,. \end{cases} \tag{1.1.i}$$

Let $\vec{\varrho}_j$ be the collection of signs obtained from $\vec{\varrho}$ by changing the i^{th} sign where $i = 2j$ or $i = 2j-1$. We may then use Equation (1.1.i) to see that $\gamma_j W_{\vec{\varrho}} = W_{\vec{\varrho}_j}$. Since we can pass from any given collection of signs $\vec{\varrho}$ to any other collection of signs $\vec{\varepsilon}$ by changing entries sequentially,

$$\dim W_{\vec{\varrho}} = \dim W_{\vec{\varepsilon}}$$

for any collection of signs $\vec{\varrho}$ and $\vec{\varepsilon}$. This shows that

$$\dim(W) = 2^{\bar{m}} \dim(W_{\vec{\varrho}})\,.$$

Consequently, the dimension of **any** Clifford module must be divisible by $2^{\bar{m}}$. This shows that W is irreducible if $\dim(W) = 2^{\bar{m}}$.

Let $\{w_1, ..., w_\ell\}$ be a basis for $W_{(1,\ldots,1)}$. We consider tuples of odd integers $J = (j_1, ..., j_p)$ for $1 \le j_1 < ... < j_p < m$. Define

$$w_\mu^J := \gamma_{j_1} ... \gamma_{j_p} w_\mu \quad \text{for} \quad 1 \le \mu \le \ell\,.$$

We may then express

$$W_{\vec{\varrho}} = \operatorname{Span}_\mu \{w_\mu^{J_{\vec{\varrho}}}\} \quad \text{where} \quad J_{\vec{\varrho}} := \{2j-1 : \vec{\varrho}_j = -1\}\,.$$

Thus $\{w_\mu^J\}_{J,\mu}$ is a basis for W. We may now decompose

$$W = \oplus_{\mu=1}^{\ell} W_\mu \quad \text{where} \quad W_\mu := \operatorname{Span}_J \{w_\mu^J\}\,. \tag{1.1.j}$$

Clearly $\gamma_j W_\mu \subset W_\mu$ for j odd. Furthermore, since $\gamma_{2i} = \sqrt{-1}\gamma_{2i-1}\tau_i$, we also have $\gamma_j W_\mu \subset W_\mu$ for $j = 2i$ even. The W_μ are Clif (V) modules of dimension $2^{\bar{m}}$ and thus, as noted above, must be irreducible. We have

$$\gamma_j w_\mu^J = \varepsilon(j, J) w_\mu^{K(j,J)} \quad \text{for} \quad 1 \le \mu \le \ell,$$

where $\varepsilon(j, J) = \pm 1$ is an appropriately chosen sign and where $K(j, J)$ is an appropriately chosen multi-index. Thus the actions of the endomorphisms γ_j on these spaces are all isomorphic and equivalent. The Lemma now follows from Equation (1.1.j). $\square$

If $\dim(V)$ is odd, then there are two inequivalent complex representations of $\text{Clif}\,(V)$.

Lemma 1.1.6 *Let $m := \dim(V) = 2\bar{m} + 1$ be odd. There exist two inequivalent irreducible complex Clifford modules $\mathcal{S}_\pm$ for V so that if W is any complex Clifford module for V, then $W = \ell_+\mathcal{S}_+ \oplus \ell_-\mathcal{S}_-$.*

Proof: Let γ define a $\text{Clif}\,(V)$ module structure on W. Fix an orientation for V and let $\{e_1, ..., e_m\}$ be an oriented orthonormal basis for V. Let

$$\gamma_i := \gamma(e_i) \in \text{End}\,(V)\,.$$

Define the *normalized orientation class* by setting

$$\mathfrak{orn} := (\sqrt{-1})^{\bar{m}+1} e_1 * ... * e_m \in \text{Clif}\,(V)\,.$$

We then have

$$\mathfrak{orn} * \mathfrak{orn} = 1 \quad \text{and} \quad e_i * \mathfrak{orn} = \mathfrak{orn} * e_i \text{ for } 1 \leq i \leq m\,.$$

We use $\gamma(\mathfrak{orn})$ to decompose $W = W^+ \oplus W^-$ where

$$W^\pm := \{w \in W : \gamma(\mathfrak{orn})w = \pm w\}\,.$$

Since

$$\gamma_i \cdot \gamma(\mathfrak{orn}) = \gamma(e_i * \mathfrak{orn}) = \gamma(\mathfrak{orn} * e_i) = \gamma(\mathfrak{orn}) \cdot \gamma_i,$$

γ_i preserves the eigenspaces of $\mathfrak{orn}$ and hence

$$\gamma_i W^\pm = W^\pm \quad \text{for} \quad 1 \leq i \leq m\,.$$

Therefore W^+ and W^- are also $\text{Clif}\,(V)$ modules. Let

$$V_0 := \text{Span}\,\{e_1, ..., e_{2\bar{m}}\}\,.$$

By Lemma 1.1.5 we may decompose

$$W^+ = \oplus_\mu W^+_\mu \quad \text{and} \quad W^- = \oplus_\nu W^-_\nu$$

as the direct sum of irreducible $\text{Clif}\,(V_0)$ modules. We then have

$$\gamma_{2\bar{m}+1} = \sqrt{-1}^{\bar{m}+1} \gamma_1 \cdots \gamma_m \cdot \begin{cases} -\text{Id} & \text{on } W^+_\mu, \\ +\text{Id} & \text{on } W^-_\nu\,. \end{cases}$$

This implies that

$$\gamma_{2\bar{m}+1} W^+_\mu \subset W^+_\mu \quad \text{and} \quad \gamma_{2\bar{m}+1} W^-_\nu \subset W^-_\nu\,.$$

Consequently these subspaces are also irreducible $\text{Clif}\,(V)$ modules. They are inequivalent as $\gamma(\mathfrak{orn}) = +1$ on W^+_μ and $\gamma(\mathfrak{orn}) = -1$ on W^-_ν. This gives the decomposition described in the Lemma. $\square$

Lemmas 1.1.5 and 1.1.6 give the structure of the *complex Clifford modules*; this structure is periodic mod 2 and follows from the isomorphism

$$\text{Clif}\,(V) \otimes_{\mathbb{R}} \mathbb{C} \approx \begin{cases} M_{2^{\bar{m}}}(\mathbb{C}) & \text{if } \dim(V) = 2\bar{m}, \\ M_{2^{\bar{m}}}(\mathbb{C}) \oplus M_{2^{\bar{m}}}(\mathbb{C}) & \text{if } \dim(V) = 2\bar{m} + 1\,. \end{cases}$$

The corresponding structure of the real Clifford modules and of the quaternionic Clifford modules is periodic mod 8 and is a bit more complicated to describe. We refer to Karoubi [252] for further details.

1.1.5 Clifford bundles

Let (M, g) be a Riemannian manifold. Let W be an auxiliary complex vector bundle over M. We say that

$$\gamma : TM \to \operatorname{End}(W)$$

defines a *Clifford module structure* on W if the Clifford commutation relations

$$\gamma(x)\gamma(y) + \gamma(y)\gamma(x) = -2g(x, y)\mathrm{Id}_W \quad \forall x, y \in TM$$

are satisfied. We can also regard γ as defining a map from T^*M to $\operatorname{End}(W)$ as well. We say that γ is a *unitary Clifford module structure* if W is equipped with a Hermitian inner product $(\cdot, \cdot)$ and if

$$(\gamma(x)w_1, w_2) + (w_1, \gamma(x)w_2) = 0 \quad \forall x \in TM,\ w_1 \in W,\ w_2 \in W\,.$$

If ∇ is a connection on W, we say that ∇ is a *unitary connection* if

$$(\nabla_x w_1, w_2) + (w_1, \nabla_x w_2) = x(w_1, w_2) \quad \forall x \in TM,\ w_1 \in W,\ w_2 \in W\,.$$

Define $\nabla\gamma : TM \otimes TM \to \operatorname{End}(W)$ by setting

$$(\nabla_x\gamma)(y)w := \nabla_x\{\gamma(y)w\} - \gamma(\nabla_x y)w - \gamma(y)\nabla_x w\,.$$

We say that ∇ is a *compatible connection* if $\nabla\gamma = 0$.

Lemma 1.1.7 *Let (M, g) be a Riemannian manifold. Let W be a vector bundle over M. Let γ be a Clifford module structure on W.*

1. *Let $\{e_1, ..., e_m\}$ be a local orthonormal frame for TM over a contractible open subset U of M. Then there exists a local frame $\vec{s}$ for W over U so the matrices $\gamma(e_i)^{\vec{s}}$ are constant.*
2. *There exists a Hermitian inner product $(\cdot, \cdot)$ on W so that γ is unitary.*
3. *There exists a compatible unitary connection ∇ on W.*

Proof: To prove the first assertion, we suppose m is even; the argument is similar if m is odd. The given orthonormal frame $\{e_1, ..., e_m\}$ for TM identifies

$$TM|_U = \mathbb{R}^m \times U \quad \text{and} \quad \operatorname{Clif}(M) = \operatorname{Clif}(\mathbb{R}^m) \times U\,.$$

Let the joint eigenspaces $W_{\vec{\varrho}}$ be defined as in the proof of Lemma 1.1.5. Since these eigenspaces have constant rank, they patch together to define smooth vector bundles over U. Since U is contractible, $W_{(1,\ldots,1)}$ is trivial. Thus we can choose local frames $\{w_1, ..., w_\ell\}$ for this bundle. The proof of Lemma 1.1.5 then shows the Clifford module structure is locally a product

$$W|_U = \ell \cdot \mathcal{S} \times U.$$

Assertion (1) now follows. Since the matrices in question are skew-symmetric with respect to the given basis, Hermitian inner products exist locally on W for which γ will be unitary. Since γ will be unitary with respect to a convex combination of such inner products, we can use a partition of unity to establish Assertion (2).

Let $\omega^{\vec{s}}$ be the *connection* 1 *form* of ∇ relative to a local frame for W constructed above. Let $\gamma_{j;i}$ be the components of $\nabla\gamma$. Then

$$\gamma_{j;i}^{\vec{s}} = e_i\gamma_j^{\vec{s}} + \omega_i^{\vec{s}}\gamma_j^{\vec{s}} - \gamma_j^{\vec{s}}\omega_i^{\vec{s}} - \Gamma_{ijk}\gamma_k^{\vec{s}}\,. \tag{1.1.k}$$

Thus the convex combination of compatible unitary connections is again a compatible unitary connection. Consequently it suffices to work locally in order to establish the third assertion.

We choose the frame so that $e_i\gamma_j^{\vec{s}} = 0$. Define the *spin connection* by setting

$$\omega_i^{\vec{s}} := \tfrac{1}{4}\Gamma_{ikl}\gamma_k^{\vec{s}}\gamma_l^{\vec{s}}\,.$$

The relation $\Gamma_{ijk} = -\Gamma_{ikj}$ and the Clifford commutation rules imply

$$\begin{aligned}
[\omega_i^{\vec{s}},\gamma_j^{\vec{s}}] - \Gamma_{ijk}\gamma_k^{\vec{s}} &= \tfrac{1}{4}\Gamma_{ikl}(\gamma_k^{\vec{s}}\gamma_l^{\vec{s}}\gamma_j^{\vec{s}} - \gamma_j^{\vec{s}}\gamma_k^{\vec{s}}\gamma_l^{\vec{s}}) - \Gamma_{ijk}\gamma_k^{\vec{s}}\\
&= \tfrac{1}{4}\Gamma_{ikl}(-2\delta_{lj}\gamma_k^{\vec{s}} + 2\delta_{kj}\gamma_l^{\vec{s}}) - \Gamma_{ijk}\gamma_k^{\vec{s}}\\
&= \Gamma_{ijl}\gamma_l^{\vec{s}} - \Gamma_{ijk}\gamma_k^{\vec{s}} = 0\,.
\end{aligned}$$

Consequently, ∇ is compatible. Furthermore, as γ is unitary, $\omega_i^{\vec{s}}$ is skew-symmetric so ∇ is unitary as well. This establishes Assertion (3). □

Let γ give W a $\mathrm{Clif}\,(M)$ module structure. We define a *tangential Clifford module structure* on $W|_{\partial M}$ by setting

$$\gamma_a^T := -\gamma_m\gamma_a\,. \tag{1.1.l}$$

This satisfies the Clifford commutation rules since

$$\gamma_a^T\gamma_b^T + \gamma_b^T\gamma_a^T = \gamma_m\gamma_a\gamma_m\gamma_b + \gamma_b\gamma_m\gamma_a\gamma_m = \gamma_a\gamma_b + \gamma_b\gamma_a = -2\delta_{ab}\mathrm{Id}\,.$$

This structure will play a central role in our discussion of spectral boundary conditions subsequently. If ∇ is a connection on W, then

$$\gamma_{a:b}^T := -\nabla_{e_b}\gamma_m\gamma_a + \gamma_m\gamma_a\nabla_{e_b} + \Gamma_{abc}\gamma_m\gamma_c\,.$$

Suppose that $\nabla^M\gamma = 0$ so $\nabla_{e_i}\gamma_j - \gamma_j\nabla_{e_i} - \Gamma_{ijk}e_k = 0$. Consequently,

$$\begin{aligned}
\gamma_{a:b}^T &= -\Gamma_{bmi}\gamma_i\gamma_a - \Gamma_{bai}\gamma_m\gamma_i + \Gamma_{bac}\gamma_m\gamma_c \qquad (1.1.\mathrm{m})\\
&= -\Gamma_{bmc}\gamma_c\gamma_a - \Gamma_{bam}\gamma_m\gamma_m = L_{bc}\gamma_c\gamma_a + L_{ab}\mathrm{Id}\,.
\end{aligned}$$

Thus $\nabla^{\partial M}\gamma^T$ is non-zero in general. We do, however, have that

$$\gamma_{a:a}^T = 0\,. \tag{1.1.n}$$

1.1.6 Duality

We shall let V^* denote the *dual vector bundle*. In studying the heat content asymptotics, it is natural to regard the initial temperature ϕ as a section to V and to regard the specific heat ρ as a section to V^*. The local heat energy density is then given by the *dual pairing* between V and V^*; we shall denote this pairing by $\langle\phi,\rho\rangle$. If ∇ is a connection on V, let $\tilde\nabla$ be the *dual connection* on V^*. Similarly, if E is an endomorphism of V, let $\tilde E$ be the *dual endomorphism* of V^*. Then $\tilde\nabla$ and $\tilde E$ are characterized by the identities

$$\langle\nabla\phi,\rho\rangle+\langle\phi,\tilde\nabla\rho\rangle=d\langle\phi,\rho\rangle\quad\text{and}\quad\langle E\phi,\rho\rangle=\langle\phi,\tilde E\rho\rangle\,. \tag{1.1.o}$$

We clear the previous notation. Let $\omega^{\vec s}$ be the connection 1 form for ∇ relative to a local frame $\vec s$ for V. Then the connection 1 form for $\tilde\nabla$ on V^* relative to the dual frame $\tilde{\vec s}$ for $\tilde W$ is $-\tilde\omega^{\vec s}$. This means that

$$\nabla_{\partial^x_\mu}=\partial^x_\mu+\omega^{\vec s}_\mu\quad\text{and}\quad\tilde\nabla_{\partial^x_\mu}=\partial^x_\mu-\tilde\omega^{\vec s}_\mu.$$

If V admits an inner product $(\cdot,\cdot)$, then the inner product defines a conjugate linear isomorphism ψ between V and V^* so that

$$\langle\phi,\rho\rangle=(\phi,\psi\rho)\,. \tag{1.1.p}$$

It is convenient for the most part, except in discussing questions of self-adjointness, not to introduce the auxiliary structure $(\cdot,\cdot)$ and to work instead with V^*.

We say that a connection ∇ on V is a *unitary connection* and that an endomorphism E of V is *self-adjoint* if and only if

$$(\nabla\phi_1,\phi_2)+(\phi_1,\nabla\phi_2)=d(\phi_1,\phi_2)\quad\text{and}\quad(E\phi_1,\phi_2)=(\phi_1,E\phi_2)\,,$$

respectively. For example, we use Equation (1.1.a) to see that the Levi-Civita connection is unitary.

If P is a partial differential operator on $C^\infty(V)$, we let $\tilde P$ be the *dual operator* on $C^\infty(V^*)$. This operator is characterized by the relation

$$\int_M\langle P\phi,\rho\rangle dx=\int_M\langle\phi,\tilde P\rho\rangle dx\quad\forall\phi\in C_0^\infty(V),\rho\in C_0^\infty(V^*)\,. \tag{1.1.q}$$

1.1.7 Volume of spheres

In Sections 1.9, 3.8, and 3.9, we will express certain universal coefficients in terms of the *volume of spheres*. We therefore recall the following classical formulae.

Lemma 1.1.8 $\operatorname{vol}(S^{2j})=\frac{j!\pi^j2^{2j+1}}{2j!}$ *and* $\operatorname{vol}(S^{2j-1})=\frac{2\pi^j}{(j-1)!}$.

Proof: We use polar coordinates and integrate by parts to see:

$$\begin{aligned}\pi^j\sqrt\pi&=\int_{\mathbb R^{2j+1}}e^{-|x|^2}dx=\int_0^\infty\int_{S^{2j}}r^{2j}e^{-r^2}d\theta dr\\&=\operatorname{vol}(S^{2j})\int_0^\infty r^{2j}e^{-r^2}dr=\operatorname{vol}(S^{2j})\tfrac{2j-1}{2}\tfrac{2j-3}{2}\ldots\tfrac12\int_{r=0}^\infty e^{-r^2}dr\,.\end{aligned}$$

We may solve this equation to determine $\mathrm{vol}\,(S^{2j})$. Similarly, we compute:

$$\begin{aligned}
\pi^j &= \int_{\mathbb{R}^{2j}} e^{-|x|^2} dx = \int_{r=0}^{\infty} \int_{S^{2j-1}} r^{2j-1} e^{-r^2} d\theta dr \\
&= \mathrm{vol}\,(S^{2j-1}) \int_0^{\infty} r^{2j-1} e^{-r^2} dr = \mathrm{vol}\,(S^{2j-1})(j-1)! \int_0^{\infty} r e^{-r^2} dr \\
&= \tfrac{1}{2}\mathrm{vol}\,(S^{2j-1})(j-1)!\,.
\end{aligned}$$

We solve for $\mathrm{vol}\,(S^{2j-1})$ to complete the proof. $\square$

1.2 The geometry of operators of Laplace type

1.2.1 The symbol of an operator

Let V be a smooth vector bundle over a compact Riemannian manifold M. Let $\vec{a} = (a_1, ..., a_m)$ be an m-tuple of non-negative integers, let $e = (e_1, ..., e_m)$ be a local orthonormal frame for TM, let $x = (x_1, ..., x_m)$ be a system of local coordinates on M, and let $\xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m$. We fix a connection ∇ on V and define

$$\begin{aligned}
&\nabla^{\vec{a}} := (\nabla_{e_1})^{a_1} ... (\nabla_{e_m})^{a_m}, \quad &&\partial_{\vec{a}}^x := (\partial_1^x)^{a_1} ... (\partial_m^x)^{a_m}, \\
&|\vec{a}| = a_1 + ... + a_m, &&\xi^{\vec{a}} = \xi_1^{a_1} ... \xi_m^{a_m}\,.
\end{aligned}$$

Let $P : C^\infty(V) \to C^\infty(V)$ be a partial differential operator on V of order d. We shall always assume $d > 0$. We may expand

$$P := \sum_{|\vec{a}| \le d} p_{\vec{a}}(x) \nabla^{\vec{a}}$$

where the $p_{\vec{a}}$ are smooth sections to the bundle of endomorphisms of V. The *leading symbol* $\sigma_L(P)(\xi)$ is defined by replacing ∇_{e_i} by $\sqrt{-1}\xi_i$ and suppressing the lower order terms. More precisely,

$$\sigma_L(P)(\xi) := (\sqrt{-1})^d \sum_{|\vec{a}| = d} p_{\vec{a}}(x) \xi^{\vec{a}}\,.$$

In principle, we should also include the order d in the notation as if P is an operator of order d, and if we were to think of P as an operator of order $d + 1$, then the leading symbol would vanish. In practice, this ambiguity causes no confusion and we shall suppress d from the notation in the interest of simplicity.

If $\omega \in T^*M$, then we may expand $\omega = \xi_i(\omega) e^i$. The association which sends ω to $\sigma_L(P)(\xi(\omega))$ induces a well-defined map

$$\sigma_L(P) : T^*M \to \mathrm{End}\,(V)$$

which is independent of both the choice of the local orthonormal frame e and of the choice of the connection ∇. We have the *symbol composition rule*

$$\sigma_L(P \circ Q) = \sigma_L(P) \circ \sigma_L(Q)\,.$$

If V is equipped with a positive definite Hermitian metric, let P^* be the formal adjoint. The *adjoint symbol* is then

$$\sigma_L(P^*) = \sigma_L(P)^*\,.$$

We can also define the leading symbol more directly. If ξ is a cotangent vector at x_0, choose $f \in C^\infty(M)$ so that $f(x_0) = 0$ and so that $df(x_0) = \xi$. If $v_0 \in V_{x_0}$, choose $v \in C^\infty(V)$ so that $v(x_0) = v_0$. We then have

$$\sigma_L(P)(\xi)v_0 = \lim_{t\to\infty}\{t^{-d}e^{-\sqrt{-1}tf}P(e^{\sqrt{-1}tf}v)\}(x_0)\,.$$

This is independent of the particular function f and of the particular section v which were chosen and is invariantly defined on the cotangent bundle.

Recall that a second order partial differential operator D on $C^\infty(V)$ is said to be of *Laplace type* if locally D has the form

$$D = -(g^{\mu\nu}\mathrm{Id}\,\partial_\mu^x\partial_\nu^x + a^\mu\partial_\mu^x + b)\,. \tag{1.2.a}$$

It is then immediate from the definition that D is of Laplace type if and only if $\sigma_L(D)(\xi) = |\xi|^2\mathrm{Id}$.

A first order operator $A = \gamma^\mu\partial_\mu + \phi$ is said to be of *Dirac type* if and only if A^2 is of Laplace type, i.e. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}\mathrm{Id}$ so γ defines a *Clifford module structure* on the vector bundle in question.

1.2.2 An invariant representation of an operator of Laplace type

Let V be a smooth vector bundle over M and let D be an operator of Laplace type on $C^\infty(V)$. The coordinate formalism of Equation (1.2.a) is not invariant and it is useful to express D in terms of geometric objects. Let ∇ be a connection on V. We use this connection and the Levi-Civita connection of M to covariantly differentiate tensors of all types; let ";" denote the components of multiple covariant differentiation. Let E be an endomorphism of V. We define the associated operator $D(\nabla, E)$ of Laplace type by setting

$$D(\nabla, E)\phi := -(\phi_{;ii} + E\phi)\,, \tag{1.2.b}$$

where the tensor $\phi_{;jk}$ is defined by Equation (1.1.f). This is also denoted symbolically by

$$D(\nabla, E)\phi = -(\mathrm{Tr}\,(\nabla^2)\phi + E\phi)\,.$$

In terms of a coordinate frame, this takes the form

$$D = -(g^{\mu\nu}\nabla_{\partial_\mu^x}\nabla_{\partial_\nu^x} - g^{\mu\nu}\Gamma_{\mu\nu}{}^\sigma\nabla_{\partial_\sigma^x} + E)\,.$$

Lemma 1.2.1 *Let D be an operator of Laplace type on a Riemannian manifold (M, g). There exists a unique connection ∇ on V and a unique endomorphism E on V so that $D = D(\nabla, E)$. If D locally has the form*

$$D = -(g^{\mu\nu}\mathrm{Id}\,\partial_\mu^x\partial_\nu^x + a^\mu\partial_\mu^x + b)\,,$$

then the connection 1 *form* ω *of* ∇ *and the endomorphism* E *are given by*

$$\begin{aligned}\omega_\nu &= \tfrac{1}{2}(g_{\nu\mu}a^\mu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon\nu}\mathrm{Id}\,)\\ E &= b - g^{\nu\mu}(\partial_\nu^x\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma)\,.\end{aligned}$$

Proof: Fix a local frame for V. We have dually by Equation (1.1.a) that

$$\nabla(dx^\nu) = -\Gamma_{\mu\sigma}{}^\nu dx^\mu \otimes dx^\sigma\,.$$

We expand the covariant derivative operator in the form

$$\begin{aligned}&\nabla = dx^\mu \otimes (\partial_\mu^x + \omega_\mu) \quad \text{so}\\ &\nabla^2 = dx^\mu \otimes dx^\sigma \otimes \{-\Gamma_{\mu\sigma}{}^\nu(\partial_\nu^x + \omega_\nu) + (\partial_\mu^x + \omega_\mu)(\partial_\sigma^x + \omega_\sigma)\}\,.\end{aligned}$$

Thus, using Equation (1.2.b), we have that

$$\begin{aligned}D(\nabla, E) \;=\; &-g^{\mu\sigma}\{\partial_\mu^x\partial_\sigma^x + 2\omega_\mu\partial_\sigma^x - \Gamma_{\mu\sigma}{}^\nu\partial_\nu^x\\ &+\partial_\mu^x(\omega_\sigma) + \omega_\mu\omega_\sigma - \Gamma_{\mu\sigma}{}^\nu\omega_\nu\} - E\,.\end{aligned}$$

Consequently $D = D(\nabla, E)$ precisely when we have the identities

$$\begin{aligned}a^\nu &= 2g^{\nu\mu}\omega_\mu - g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu\mathrm{Id}\,, \quad \text{and}\\ b &= g^{\mu\sigma}(\partial_\mu^x\omega_\sigma + \omega_\mu\omega_\sigma - \Gamma_{\mu\sigma}{}^\nu\omega_\nu) + E\,.\end{aligned} \tag{1.2.c}$$

We solve for ω and E to establish the uniqueness assertion of the Lemma.

Using Equation (1.2.c) as the definition then permits us to establish a local existence result simply by reversing the steps in the computation; the uniqueness assertion then implies the connections and endomorphisms which are defined locally using Equation (1.2.c) patch together properly. $\square$

1.2.3 The Dual Operator

Let V be a smooth vector bundle over a Riemannian manifold M and let $\langle\cdot,\cdot\rangle$ denote the pairing between V and the dual bundle V^*. Let dx be the Riemannian measure. If $\phi \in C^\infty(V)$ and if $\rho \in C^\infty(V^*)$, then we define

$$\langle\phi,\rho\rangle_{L^2} := \int_M \langle\phi,\rho\rangle(x)dx\,.$$

We say that V is *Hermitian* if V is equipped with a positive definite Hermitian innerproduct $(\cdot,\cdot)$; in this setting, we shall define

$$(\phi_1,\phi_2)_{L^2} := \int_M (\phi_1,\phi_2)(x)dx \quad \text{for} \quad \phi_i \in C^\infty(V)\,.$$

Let $C_0^\infty(V)$ be the space of smooth sections to V which are compactly supported in the interior of M; this means that ϕ vanishes on some neighborhood of ∂M. Let D be an operator of Laplace type on $C^\infty(V)$. The *dual operator* $\tilde{D}$ is the operator of Laplace type on $C^\infty(V^*)$ characterized by the identity

$$\langle D\phi,\rho\rangle_{L^2} = \langle\phi,\tilde{D}\rho\rangle_{L^2} \quad \forall \quad \phi \in C_0^\infty(V) \quad \text{and} \quad \rho \in C_0^\infty(V^*)\,.$$

The connection and endomorphism which are associated to $\tilde{D}$ are the *dual endomorphism of* D and the *dual connection of* D, i.e.

Lemma 1.2.2 *Let* $D = D(\nabla, E)$ *be an operator Laplace type on a vector bundle* V. *Let* $\tilde{\nabla}$ *be the dual connection on* V^* *and let* $\tilde{E}$ *be the dual endomorphism on* V^*. *Then* $\tilde{D} = D(\tilde{\nabla}, \tilde{E})$.

Proof: Since the support of ϕ and ρ is disjoint from the boundary, there are no boundary correction terms. By Lemma 1.4.17,

$$\begin{aligned}\langle \phi, \tilde{D}\rho\rangle_{L^2} &= \langle D\phi, \rho\rangle_{L^2} = -\int_M \langle \phi_{;ii} + E\phi, \rho\rangle dx \\ &= -\int_M \langle \phi, \rho_{;ii} + \tilde{E}\rho\rangle dx\,.\end{aligned}$$

This implies that $\tilde{D}\rho = -(\rho_{;ii} + \tilde{E}\rho)$; note that we are using $\tilde{\nabla}$ to covariantly differentiate ρ. The Lemma now follows from the uniqueness assertions in Lemma 1.2.1. □

Suppose that V has a Hermitian inner product. We use the conjugate linear isomorphism ψ discussed in Equation (1.1.p) to identify V and V^*. Then ∇ is unitary if and only if $\nabla = \tilde{\nabla}$, E is self-adjoint if and only if $E = \tilde{E}$, and D is formally self-adjoint if and only if $\tilde{D} = D$. The following observation is then immediate from Lemmas 1.2.1 and 1.2.2:

Lemma 1.2.3 *Let* D *be an operator Laplace type on a Hermitian vector bundle* V. *Then* $D = D(\nabla, E)$ *is formally self-adjoint if and only if* ∇ *is unitary and* E *is self-adjoint.*

1.2.4 The form valued Laplacian

Let $\Lambda(M)$ be *exterior algebra* bundle on the cotangent bundle of M. Choose a local system of coordinates $x = (x_1, ..., x_m)$ on M. If we have a multi-index $I = \{1 \le \mu_1 < ... < \mu_p \le m\}$, set

$$dx^I := dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}\,.$$

The *exterior derivative* $d : C^\infty(\Lambda(M)) \to C^\infty(\Lambda(M))$ is defined by

$$d\Big(\sum_I f_I dx^I\Big) := \sum_{\mu,I} \partial_\mu^x(f_I) dx^\mu \wedge dx^I\,. \tag{1.2.d}$$

Let $\delta : C^\infty(\Lambda(M)) \to C^\infty(\Lambda(M))$ be the adjoint operator; this is the *interior derivative.* Let

$$\Delta := d\delta + \delta d$$

be the *form valued Laplacian.* We may use the grading on the exterior algebra to decompose

$$\Delta = \oplus_p \Delta^p$$

where Δ^p acts on $C^\infty(\Lambda^p(M))$. We will also use the notation Δ_M and Δ_M^p occasionally if it is necessary to include the underlying manifold M explicitly in the notation. We will also occasionally delete the superscript "0" and let $\Delta = \Delta^0$ be *the scalar Laplacian* when the context is clear.

The *Hodge-de Rham isomorphism* shows that $\ker \Delta$ has topological significance. Let $H^p(M;\mathbb{R})$ denote the real cohomology groups of M.

Theorem 1.2.4 (Hodge-de Rham) *Let M be a closed Riemannian manifold. There is a natural isomorphism* $\ker \Delta^p = H^p(M;\mathbb{R})$.

We may use the *Weitzenböck formula* to express Δ in the form given in Lemma 1.2.1. Let $\mathfrak{e}(\xi)$ denote (left) exterior multiplication by a cotangent vector ξ acting on the bundle $\Lambda(M)$. This means that

$$\mathfrak{e}(\xi)\omega := \xi \wedge \omega\,.$$

Let $\mathfrak{i}(\xi) := \mathfrak{e}(\xi)^*$ be the dual map (left) interior multiplication, and let

$$\gamma(\xi) := \mathfrak{e}(\xi) - \mathfrak{i}(\xi)\,.$$

We use the metric to identify the tangent and cotangent spaces. If $\{e_i\}$ is a local orthonormal frame for $TM = T^*M$, we set

$$\mathfrak{e}_i := \mathfrak{e}(e_i), \quad \mathfrak{i}_i := \mathfrak{i}(e_i), \quad \text{and} \quad \gamma_i := \gamma(e_i)\,.$$

If $1 \le i_1 < ... < i_p \le m$, then we compute

$$\begin{aligned} \mathfrak{e}_1\{e_{i_1} \wedge ... \wedge e_{i_p}\} &= \begin{cases} e_1 \wedge e_{i_1} \wedge ... \wedge e_{i_p} & \text{if} \quad 1 < i_1, \\ 0 & \text{otherwise}, \end{cases} \\ \mathfrak{i}_1\{e_{i_1} \wedge ... \wedge e_{i_p}\} &= \begin{cases} e_{i_2} \wedge ... \wedge e_{i_p} & \text{if} \quad 1 = i_1, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{1.2.e}$$

It is now clear that γ defines a (left) Clifford module structure on the exterior algebra since the *Clifford commutation relations* are satisfied, i.e.

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\mathrm{Id} \quad \text{so} \quad \gamma(\xi)^2 = -|\xi|^2\mathrm{Id}\ .$$

By Equation (1.2.d), the leading symbol of d is given by

$$\sigma_L(d)(\xi) = \sqrt{-1}\mathfrak{e}(\xi)\,.$$

Dually, the leading sybol of δ is given by

$$\sigma_L(\delta)(\xi) = -\sqrt{-1}\mathfrak{i}(\xi)\,.$$

Consequently, the leading symbol of $d+\delta$ is given by $\sqrt{-1}\gamma$. Because we have $\gamma(\xi)^2 = -|\xi|^2\mathrm{Id}$, Δ is an operator of Laplace type and $d+\delta$ is an operator of Dirac type.

Let Ω_{ij} be the curvature of the Levi-Civita connection acting on the exterior algebra. We refer to [189] Section 4.1 for the proof of the following result which gives the *endomorphism* and the *connection* defined by the form valued Laplacian.

Lemma 1.2.5 *Let $\Delta := d\delta + \delta d$ be the form valued Laplacian.*

1. $d = \mathfrak{e}_i\nabla_{e_i}$ *and* $\delta = -\mathfrak{i}_i\nabla_{e_i}$.

2. *We have* $\Delta = D(\nabla, E)$ *where*

   (a) ∇ *is the Levi-Civita connection.*

   (b) $E = -\frac{1}{2}\gamma_i\gamma_j\Omega_{ij}$.

We use the decomposition of $\Lambda(M) = \oplus_p \Lambda^p(M)$ to decompose $\Delta = \oplus_p \Delta^p$ where Δ^p is the *p form valued Laplacian* on $C^\infty(\Lambda^p(M))$. The following is then an immediate consequence of Lemma 1.2.5.

Lemma 1.2.6 *Let E_p be the endomorphism and let Ω_p be the curvature defined by the p form valued Laplacian Δ^p, respectively. Then*

1. *We have* $E_0 = 0$ *and* $\Omega_0 = 0$.
2. *We have* $E_1(e_i) = -R_{ijjk}e_k$ *and* $\Omega_{1,ij}e_k = R_{ijkl}e_l$.

1.2.5 The Hodge $\star$ operator

Let $\Lambda^p(M)$ be the bundle of smooth p forms over a compact oriented Riemannian manifold. Let orn be the orientation form. The *Hodge $\star$ operator* is the linear map

$$\star : \Lambda^p(M) \to \Lambda^{m-p}(M)$$

which is characterized by the identity

$$(\omega_p, \theta_p)\text{orn} = \omega_p \wedge \star\theta_p \quad \text{for all} \quad \omega_p, \theta_p \in \Lambda^p(M)\,.$$

For example, if $\{e_1, ..., e_m\}$ is a local oriented orthonormal basis for TM, then

$$\star\{e_1 \wedge ... \wedge e_p\} = e_{p+1} \wedge ... \wedge e_m\,.$$

We refer to [189] for the proof of the following result:

Lemma 1.2.7 *1.* $\star^{m-p}\star^p = (-1)^{p(m-p)}$.

2. $\star^{m-p}d^{m-p-1}\star^{p+1} = (-1)^{mp+1}\delta^p$.
3. $\star^{m-p}\Delta^{m-p}\star^p = (-1)^{p(m-p)}\Delta^p$.

The Hodge $\star$ operator can be expressed in terms of the Clifford module structure $\gamma := \mathfrak{e} - \mathfrak{i}$ discussed previously. Let $\{e_1, ..., e_m\}$ be a local oriented orthonormal frame for TM. Set

$$\tilde\star := \varepsilon_m \gamma(e_1)...\gamma(e_m) \in \text{End}\,(\Lambda(M)) \tag{1.2.f}$$

where the 4^{th} root of unity

$$\varepsilon_m := \begin{cases} (\sqrt{-1})^{m/2} & \text{if } m \text{ is even,} \\ (\sqrt{-1})^{(m+1)/2} & \text{if } m \text{ is odd,} \end{cases}$$

has been chosen to ensure that

$$\tilde\star^2 = \text{Id}\,.$$

We then have

$$\star^p = \varepsilon_{m,p}\tilde\star^p \text{ on } \Lambda^p(M) \quad \text{and} \quad \tilde\star\mathfrak{e} = (-1)^m\mathfrak{i}\tilde\star$$

where $\varepsilon_{m,p}$ is a suitably chosen 4^{th} root of unity. Since $\nabla\tilde{\star} = 0$, Lemma 1.2.5 yields the intertwining relations

$$\tilde{\star} d = (-1)^{m-1}\delta\tilde{\star} \quad \text{and} \quad \tilde{\star}\Delta_p = \Delta_{m-p}\tilde{\star}\,. \tag{1.2.g}$$

1.2.6 The Witten Laplacian

Let $\phi \in C^\infty(M)$ be an auxiliary smooth function which is called the *dilaton*. The *Witten Laplacian* is a generalization of the standard Laplacian. One twists the exterior derivative and the coderivative to define the *Witten derivative* and the *Witten co-derivative* by setting

$$d_\phi := e^{-\phi} d e^{\phi} \quad \text{and} \quad \delta_\phi := e^{\phi}\delta e^{-\phi}\,.$$

The associated second order operator is then defined by setting

$$\Delta_\phi := d_\phi\delta_\phi + \delta_\phi d_\phi \quad \text{on} \quad C^\infty(\Lambda(M))\,.$$

We shall also sometimes use the notation $\Delta_{\phi,g}$ when it is necessary to include the metric g explicitly in the notation. Since

$$\sigma_L(d_\phi) = \sigma_L(d) \quad \text{and} \quad \sigma_L(\delta_\phi) = \sigma_L(\delta),$$

we see that Δ_ϕ is an operator of Laplace type as $\sigma_L(\Delta_\phi) = \sigma_L(\Delta)$.

This operator was introduced by Witten [362] in the context of Morse theory. This operator has been used to study quantum form valued fields which interact with the background dilaton in [204, 358]. It has also been used in supersymmetric quantum mechanics [6].

The Witten Laplacian is a 0^{th} order perturbation of the ordinary Laplacian. The *endomorphism* and *connection* of the Witten Laplacian are given by:

Lemma 1.2.8

1. $\Delta_\phi = \Delta + \phi_{;i}\phi_{;i}\cdot \mathrm{Id} + \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i)$.
2. *The Levi-Civita connection is the connection associated to* Δ_ϕ.
3. $E_\phi := -\frac{1}{2}\gamma_i\gamma_j\Omega_{ij} - \phi_{;i}\phi_{;i} - \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i)$ *is the endomorphism for* Δ_ϕ.
4. *Let* $\tilde{\star}$ *be as defined in Equation* (1.2.f). *Then* $\tilde{\star}\Delta_\phi = \Delta_{-\phi}\tilde{\star}$.

Proof: Lemma 1.2.5 (1) extends to the twisted setting to yield

$$d_\phi = \mathfrak{e}_i\nabla_{e_i} + \mathfrak{e}_i\phi_{;i} \quad \text{and} \quad \delta_\phi = -\mathfrak{i}_i\nabla_{e_i} + \mathfrak{i}_i\phi_{;i}\,.$$

We use the commutation rules $\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i = \delta_{ij}$, the fact that $\nabla\mathfrak{e} = 0$, and the fact that $\nabla\mathfrak{i} = 0$ to prove Assertion (1) by computing

$$\begin{aligned}
\Delta_{\phi,g} &= \Delta_g + \mathfrak{e}_i\nabla_{e_i}\mathfrak{i}_j\phi_{;j} + \mathfrak{i}_j\phi_{;j}\mathfrak{e}_i\nabla_{e_i} - \mathfrak{i}_i\nabla_{e_i}\mathfrak{e}_j\phi_{;j} \\
&\quad -\mathfrak{e}_j\phi_{;j}\mathfrak{i}_i\nabla_{e_i} + (\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i)\phi_{;i}\phi_{;j} \\
&= \Delta_g + (\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i - \mathfrak{i}_i\mathfrak{e}_j - \mathfrak{e}_j\mathfrak{i}_i)\phi_{;j}\nabla_{e_i} + (\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_i\mathfrak{e}_j)\phi_{;ji} + \phi_{;i}\phi_{;i} \\
&= \Delta_g + (\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_i\mathfrak{e}_j)\phi_{;ji} + \phi_{;i}\phi_{;i}.
\end{aligned}$$

This shows that the associated connection does not depend on ϕ and hence is the Levi-Civita connection by Lemma 1.2.5. Assertion (3) now follows from Lemma 1.2.5 (2b).

To prove Assertion (4), we must generalize Equation (1.2.g). Since

$$d_\phi = d + \mathfrak{e}(d\phi) \quad \text{and} \quad \delta_\phi = \delta + \mathfrak{i}(d\phi)\,,$$

Equation (1.2.g) shows

$$\tilde{\star} d_\phi \quad = \quad \tilde{\star} d + \tilde{\star} e(d\phi) = (-1)^{m-1}(\delta - \mathfrak{i}(d\phi))\tilde{\star} = (-1)^{m-1}\delta_{-\phi}\tilde{\star}\,.$$

This shows that

$$\begin{aligned} &\tilde{\star}(d_\phi + \delta_\phi)\tilde{\star} = (-1)^{p-1}\{d_{-\phi} + \delta_{-\phi}\}, \quad \text{so} \\ &\tilde{\star}\tilde{\Delta}_\phi\tilde{\star} = \Delta_{-\phi}; \end{aligned}$$

Assertion (4) now follows. □

1.2.7 Operators and Elliptic Complexes of Dirac Type

Recall that a first order partial differential operator A on $C^\infty(V)$ is of *Dirac type* if the corresponding second order operator A^2 is of Laplace type. More generally, let V_i be smooth vector bundles over M which are equipped with Hermitian structures. Let

$$A : C^\infty(V_1) \to C^\infty(V_2) \tag{1.2.h}$$

be a first order partial differential operator. Let

$$A^* : C^\infty(V_2) \to C^\infty(V_1)$$

be the formal adjoint and let

$$D_1 := A^*A \qquad \text{and} \qquad D_2 := AA^*$$

be the associated second order operators. We say that Display (1.2.h) is an *elliptic complex of Dirac type* if the associated second order operators D_i are of Laplace type.

Let $\Lambda^e(M)$ (resp. $\Lambda^o(M)$) be the bundle of differential forms of even (resp. odd) degrees over a Riemannian manifold M. The *de Rham complex*

$$(d+\delta) : C^\infty(\Lambda^e(M)) \to C^\infty(\Lambda^o(M))$$

is an elliptic complex of Dirac type. Other elliptic complexes of Dirac type include the signature, spin, Yang-Mills, and Dolbeault complexes. We refer to [189] for further details.

1.2.8 Singular Structures

Let $M := (M_+, M_-)$ be a pair of compact smooth manifolds which have a common smooth boundary

$$\Sigma := \partial M_+ = \partial M_-\,.$$

A structure $\mathfrak{S}$ over M is a pair of corresponding structures $\mathfrak{S} := (\mathfrak{S}_+, \mathfrak{S}_-)$ over the manifolds $M_\pm$. Let ν be the inward unit geodesic normal vector field near the boundary;

$$\nu_+|_\Sigma = -\nu_-|_\Sigma\,.$$

A Riemannian metric g on M is a pair $g := (g_+, g_-)$ of Riemannian metrics $g_\pm$ on $M_\pm$. We shall always assume

$$g_+|_\Sigma = g_-|_\Sigma,$$

but do not assume any matching condition on the normal derivatives.

A smooth vector bundle V over M is a pair of vector bundles $V_\pm$ over $M_\pm$; we do not assume any relationship between $V_+|_\Sigma$ and $V_-|_\Sigma$; in particular, we can consider the situation when we have $\dim V_+ \neq \dim V_-$. An operator of Laplace type D over M is a pair $D_\pm$ of operators of Laplace type on $C^\infty(V)$. We use Lemma 1.2.1 to express

$$D_+ = D(\nabla_+, E_+) \quad \text{and} \quad D_- = D(\nabla_-, E_-)\,.$$

1.2.9 The evolution equation for scalar heat flow

We motivate the heat equation we shall be considering subsequently by reviewing the usual *heat transport equations* for a Riemannian manifold (M, g). Let $u \in C^\infty(M \times [0,\infty))$ be the temperature of the manifold. Heat flows from hotter to colder areas. Since the direction of maximal increase is given by $\operatorname{grad}_g(f)$, the heat flow across the boundary of an infinitesimal region R is given by $-u_{;\nu}$ where ν is the inward unit normal vector field. Thus, after integrating by parts, we see that the infinitesimal change in the total heat content of R is given by

$$-\int_{\partial R} u_{;\nu}dy = -\int_R \Delta u dx\,. \tag{1.2.i}$$

On the other hand, the change in the total heat energy content is given by

$$\partial_t \int_R u dx = \int_R \partial_t u dx \tag{1.2.j}$$

Since R is arbitrary, we use Equations (1.2.i) and (1.2.j) to see that the evolution equation for heat transport has the form

$$\partial_t u = -\Delta u\,.$$

We refer to the discussion in Pinsky [309] for further details.

1.3 Interior ellipticity

In this section, we discuss the theory of elliptic operators on closed Riemannian manifolds. We introduce the heat trace and the heat content asymptotics.

Let V be a smooth vector bundle over a closed Riemannian manifold M. Fix a connection ∇ on V. If $\phi \in C^\infty(V)$, then the C^k *norm* of ϕ is defined invariantly by setting

$$||\phi||_{\infty,k} := \sup_{x \in M} \left\{ \sum_{j \leq k} |\nabla^j \phi| \right\}.$$

Changing the connection replaces this norm by an equivalent norm so the role of the connection is inessential.

1.3.1 Elliptic operators

We now recall some basic results concerning elliptic operators and refer to [189] for further details. Let $\sigma_L(P)$ be the leading symbol of a d^{th} order partial differential operator P as defined previously in Section 1.2.1. The *symbolic spectrum* $\text{Spec}_\sigma P$ is the union of the spectra of $\sigma_L(P)(\xi)$ for $\xi \neq 0$. More formally, one defines

$$\text{Spec}_\sigma(P) := \left\{ \lambda \in \mathbb{C} : \exists 0 \neq \xi \in T^*M : \det\{\sigma_L(P)(\xi) - \lambda \cdot \text{Id}\} = 0 \right\}.$$

We emphasize that it is crucial in this definition that we are studying **non-zero** cotangent vectors. We say that P is *elliptic* if $0 \notin \text{Spec}_\sigma(P)$, i.e. if $\sigma_L(P)(\xi)$ is invertible for every $0 \neq \xi \in T^*M$. We have by homogeneity that

$$\begin{aligned} &\sigma_L(P)(t\xi) = t^d \sigma_L(P)(\xi), \quad \text{so} \\ &t^d \text{Spec}_\sigma(P) = \text{Spec}_\sigma(P) \quad \text{for any} \quad t \neq 0\,. \end{aligned}$$

Thus, in particular, if $\text{Spec}_\sigma(P) = (0, \infty)$, then necessarily d is even.

The following assertion is immediate from the definition:

Lemma 1.3.1

1. *If D is an operator of Laplace type,* $\text{Spec}_\sigma(D) = (0, \infty)$ *and D is elliptic.*
2. *If A is an operator of Dirac type,* $\text{Spec}_\sigma(A) = \mathbb{R} - \{0\}$ *and A is elliptic.*

1.3.2 Elliptic regularity

Let P be a partial differential operator on a bundle V over a closed manifold M. Let $\phi \in L^2(V)$. Let $\tilde{P}$ be the formal adjoint on V^* as defined in Equation (1.1.q). We say that $P\phi = 0$ *in the distributional sense* on an open subset U of M if $\int_U \langle \phi, \tilde{P}\rho \rangle dx = 0$ for every $\rho \in C_0^\infty(V|_U)$. One has the following result concerning *elliptic regularity*:

Theorem 1.3.2 *Let P be a d^{th} order elliptic partial differential operator on a bundle V over a smooth manifold M. Let U be an open subset of M. If $P\phi = 0$ in the distributional sense on U, then ϕ is smooth on U.*

We suppose P elliptic henceforth. If ϕ is an eigenfunction of P corresponding to the eigenvalue λ, then we have $(P - \lambda)\phi = 0$. Since $P - \lambda$ is again elliptic, Theorem 1.3.2 implies that ϕ is smooth on the interior of M. The following result provides the *spectral theory of self-adjoint elliptic operators*.

Theorem 1.3.3 *Let P be a self-adjoint elliptic partial differential operator on a bundle V over a closed Riemannian manifold M. Then:*

1. *There exists a complete orthonormal basis $\{\phi_i\}$ for $L^2(V)$ so that $\phi_i \in C^\infty(V)$ and so that $P\phi_i = \lambda_i \phi_i$.*
2. *Order the eigenvalues so $|\lambda_1| \le |\lambda_2| \ldots$ There exists $\epsilon > 0$ and $i_0 > 0$ so $|\lambda_i| \ge i^\epsilon$ for $i \ge i_0$.*
3. *If $\operatorname{Spec}_\sigma(P) = (0, \infty)$, then only a finite number of the λ_i can be negative.*
4. *For every $k \ge 0$, there exists a constant C_k and an integer n_k so that $||\phi_i||_{\infty,k} \le C_k |\lambda_i|^{n_k}$ for $i \ge i_0$.*

Remark. The collection $\{\phi_i, \lambda_i\}$ is called a *discrete spectral resolution.*

By Theorem 1.3.3, if P is self-adjoint and if the leading symbol of P has only positive eigenvalues, then all but a finite number of eigenvalues of P are also positive. Furthermore, the estimate of Assertion (2) shows the eigenvalues grow relatively rapidly. Thus the symbolic spectrum of P controls the actual spectrum of P asymptotically as $\lambda \to \infty$. The same observation is true even in the non-self-adjoint setting. The following is a special case of a more general result. For any $\varepsilon > 0$ and $n > 0$, let $\mathcal{R}_{\varepsilon,n}$ be the closed subset of $\mathbb{C}$ which is characterized in polar coordinates by

$$\mathcal{R}_{\varepsilon,n} := \{\lambda = r(\cos\theta + \sqrt{-1}\sin\theta) : \theta \in (-\varepsilon, \varepsilon) \text{ or } r \le n\}. \tag{1.3.a}$$

This is the union of a large ball around the origin with a small wedge around the real axis which contains the set $[0, \infty)$ in its interior.

Theorem 1.3.4 *Let P be a d^{th} order partial differential operator on a closed Riemannian manifold M with $\operatorname{Spec}_\sigma(P) = (0, \infty)$. Then given $\varepsilon > 0$, there exists $n = n(\varepsilon)$ so that the L^2 spectrum of P is contained in the region $\mathcal{R}_{\varepsilon,n}$.*

If the hypotheses of Theorem 1.3.4 are satisfied, then necessarily $d \ge 2$ is even. We may apply the L^2 functional calculus to define the operator

$$e^{-tP} := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathcal{R}_{\varepsilon,n}} e^{-t\lambda}(P - \lambda)^{-1} d\lambda$$

for $t > 0$, where we orient the boundary suitably. This is now a bounded operator on L^2. If $u(x;t) = e^{-tP}\phi$ for $\phi \in C^\infty(V)$, then u is characterized by the two properties

$$\begin{aligned} &(\partial_t + P)u = 0 \quad \text{for} \quad t > 0 \quad &&\text{(evolution equation)} \\ &u|_{t=0} = \phi &&\text{(initial condition)}. \end{aligned}$$

1.3.3 Heat trace asymptotics

We summarize the relevant analytic facts which we shall need as follows; they can be derived using the Seeley calculus [339]. We shall omit the proofs in the interest of brevity. Let Tr_{V_x} denote the fiber trace and let Tr_{L^2} denote the trace in the L^2 sense. We refer to [189] for further details.

Theorem 1.3.5 *Let P be a d^{th} order partial differential operator on a closed Riemannian manifold M with* $\operatorname{Spec}_\sigma(P) = (0, \infty)$. *Then:*

1. *If $t > 0$, then the operator e^{-tP} is an infinitely smoothing operator which is of trace class on L^2. There exists a smooth kernel function $K(t, x, x_1, P)$ defining a linear map from V_{x_1} to V_x so that for $t > 0$,*

$$e^{-tP}\phi(x;t) = \int_M K(t, x, x_1, P)\phi(x_1)dx_1 \,.$$

2. *Let $F \in C^\infty(\operatorname{End}(V))$ be an auxiliary smooth endomorphism. Then*

$$\operatorname{Tr}_{L^2}\{Fe^{-tP}\} = \int_M \operatorname{Tr}_{V_x}\Big\{F(x)K(t, x, x, P)\Big\}dx \,.$$

As $t \downarrow 0$, there is a complete asymptotic expansion of the form

$$\operatorname{Tr}_{L^2}\{Fe^{-tP}\} \sim \sum_{n=0}^{\infty} t^{(n-m)/d} a_n(F, P) \,.$$

3. *The heat trace asymptotics $a_n(F, P)$ vanish for n odd. If n is even, there exist local endomorphism valued invariants $e_n(x, P)$ so that*

$$a_n(F, P) = \int_M \operatorname{Tr}_{V_x}\Big\{F(x)e_n(x, P)\Big\}dx \,.$$

Let V be a vector bundle of rank r over M. Fix a local frame $\vec{s} = (s_1, ..., s_r)$ for V. If H is an endomorphism of V, we expand

$$H(s_u) = H_{uv}s_v$$

to express $H = (H_{uv})$ as a matrix relative to the frame $\vec{s}$. Let $x = (x_1, ..., x_m)$ be a system of local coordinates on M. If P is a d^{th} order partial differential operator on V, we may expand

$$P = \sum_{|\vec{a}| \le d} p_{\vec{a}}(x, \xi)\partial_x^{\vec{a}}$$

where $p_{\vec{a}} = (p_{\vec{a},uv})$ is matrix valued. We define the *weight* by setting

$$\operatorname{weight}(\partial_x^{\vec{b}} p_{\vec{a},uv}) := d - |\vec{a}| + |\vec{b}| \,. \tag{1.3.b}$$

We can motivate this definition by considering the form valued Laplacian. If $p = 0$, then we may express the scalar Laplacian in the form

$$\Delta^0 = -g^{-1}\partial_\mu g g^{\mu\nu}\partial_\nu \,.$$

Thus the leading symbol $p_{\mu\nu} = -g^{\mu\nu}$ is homogeneous of total weight $2 - 2 = 0$ in the jets of the metric. The first order part $p_\nu := -g\partial_\mu\{gg^{\mu\nu}\}$ is homogeneous of total weight $2 - 1 = 1$ in the jets of the metric. The zeroth order part vanishes in this example. But if one considers the form valued Laplacian, the zeroth order part is linear in the 2 jets of the metric and quadratic in the 1 jets of the metric with coefficients which are smooth functions of the metric tensor. It is thus homogeneous of total weight $2-0=2$. Thus in particular, if $D=\Delta^p$ is the Laplacian on p forms, then the symbol $p_{\vec{a}}$ is homogeneous of weight $2-|\vec{a}|$ in the jets of the metric. Additional derivatives $\partial_x^{\vec{b}}$ increase the homogeneity by $|\vec{b}|$. Thus in this instance, the weight simply counts the total number of derivatives of the metric which appear.

This definition of weight is also motivated by the Seeley calculus [339] which gives explicit combinatorial formulae for the endomorphism valued invariants $e_n(x,P)$ which were defined above in Theorem 1.3.5. The following observation is then an immediate consequence of these formulae. We refer to the discussion in [189] for further details.

Lemma 1.3.6 *Let P be a d^{th} order partial differential operator on a vector bundle V over a closed Riemannian manifold M. Assume* $\operatorname{Spec}_\sigma(P)=(0,\infty)$. *Let $\vec{s}$ be a local frame for the bundle V and let x be a system of local coordinates on M. Then $e_{n,uv}(x,P)$ can be expressed universally as a polynomial of total weight n in the variables $\partial_x^{\vec{b}} p_{\vec{a},u_1v_1}$ of positive weight with coefficients which are smooth functions of the leading symbol of P.*

If P is an operator of Laplace type, then the situation simplifies considerably as we shall see in Section 1.7.3. We shall discuss the weight further at that point in the context of dimensional analysis.

We define scalar valued invariants $a_n(x,P)$ by setting

$$a_n(x,P):=\operatorname{Tr}_{V_x}\{e_n(x,P)\}\,. \tag{1.3.c}$$

If f is a scalar valued function, set

$$a_n(f,P):=a_n(f\cdot\operatorname{Id},P)=\int_M f(x)a_n(x,P)dx\,.$$

In particular,

$$\operatorname{Tr}_{L^2}\{e^{-tP}\}\sim\sum_{n=0}^{\infty}t^{(n-m)/d}\int_M a_n(x,P)dx\,.$$

1.3.4 The Mellin transform

We follow the discussion in [189]. Let $\Re(\lambda)$ and $\Im(\lambda)$ be the *real* and *imaginary* parts, respectively, of a complex number λ. Let P be an elliptic d^{th} order partial differential operator on a vector bundle V over a closed Riemannian manifold M. We suppose that P is self-adjoint and positive. We define the *zeta function* by

$$\zeta(s,F,P):=\operatorname{Tr}_{L^2}\{FP^{-s}\}\,.$$

This is well defined for $\Re(s)>>0$ and has a meromorphic extension to $\mathbb{C}$ with isolated simple poles. Let

$$\Gamma(s):=\int_0^\infty t^{s-1}e^{-t}dt$$

be the *Gamma* function. The integral converges for $s > 0$. We use the functional relation

$$s\Gamma(s) = \Gamma(s+1)$$

to construct a meromorphic extension of Γ to $\mathbb{C}$ with isolated simple poles at $s = 0, -1, \ldots$. There is an infinite product formula that can be used to show $\Gamma(s)$ never vanishes.

The zeta function and the heat trace are related by the *Mellin transform.* Let $\lambda > 0$. We then have

$$\lambda^{-s}\Gamma(s) = \lambda^{-s}\int_0^\infty (\lambda t)^{s-1}e^{-\lambda t}d\lambda t = \int_0^\infty t^{s-1}e^{-\lambda t}dt\,. \tag{1.3.d}$$

We sum over the individual eigenspaces to see

$$\Gamma(s)\zeta(s,F,P) = \int_0^\infty t^{s-1}\mathrm{Tr}_{L^2}(Fe^{-tP})dt\,.$$

We have

$$\int_0^1 \sum_{n=0}^{n_0} a_n(F,P)t^{(n-m)/d}t^{s-1}dt = \sum_{n=0}^{n_0}\frac{1}{s+\frac{n-m}{d}}a_n(F,P)\,.$$

Since P is assumed to be a positive operator, the heat trace decays exponentially as $t \to \infty$ so only the behavior near $t = 0$ contributes to the pole structure. Thus

$$\Gamma(s)\zeta(s,F,P) = \sum_{n=0}^{n_0}\frac{1}{s+\frac{n-m}{d}}a_n(F,P) + R_n(s)$$

where the remainder R_n is holomorphic for $\Re(s) \geq \frac{m-n_0}{d}$. This establishes the following Lemma which relates the residues of the poles of the function $\Gamma(s)\zeta(s,F,P)$ and the heat trace coefficients:

Lemma 1.3.7 *Let P be an elliptic d^{th} order positive self-adjoint partial differential operator on a closed Riemannian manifold M. Then*

$$a_n(F,P) = \mathrm{res}_{\,s=\frac{m-n}{d}}\left\{\Gamma(s)\zeta(s,F,P)\right\}.$$

Let A be a self-adjoint operator of Dirac type with $\ker(A) = \{0\}$; the associated operator A^2 is of Laplace type. There is an asymptotic expansion

$$\mathrm{Tr}_{L^2}\{FAe^{-tA^2}\} \sim \sum_{n=0}^\infty a_n^\eta(F,A)t^{(n-m-1)/2}\,.$$

The indexing convention is chosen so that the a_n^η is given by integrating a local invariant over M which is homogeneous of weight n in the derivatives of the symbol of A. The *eta function* is defined by setting

$$\eta(s,F,A) := \mathrm{Tr}_{L^2}(FA(A^2)^{-(s+1)/2})\,.$$

Equation (1.3.d) implies that

$$\Gamma(\tfrac{s+1}{2})\lambda(\lambda^2)^{-\frac{s+1}{2}} = \int_0^\infty \lambda t^{\frac{s+1}{2}-1} e^{-t\lambda^2} dt$$

and consequently, by summing over the eigenspaces,

$$\Gamma(\tfrac{s+1}{2})\eta(s,F,A) = \int_0^\infty t^{\frac{s+1}{2}-1} \mathrm{Tr}_{L^2}\left\{FAe^{-tA^2}\right\} dt\,.$$

Again, expanding

$$\int_0^1 \sum_{n=0}^{n_0} a_n^\eta(F,A) t^{\frac{n-m-1}{2}} t^{\frac{s+1}{2}-1} dt = \sum_{n=0}^{n_0} a_n^\eta(F,A)\tfrac{2}{s+n-m} \quad \text{yields}$$

$$\Gamma(\tfrac{s+1}{2})\eta(s,F,A) = \sum_{n=0}^{n_0} a_n^\eta(F,A)\tfrac{2}{s+n-m} + R_a^\eta(s)$$

where $R_a^\eta(s)$ is holomorphic for $\Re(s) \geq m-n$. This proves:

Lemma 1.3.8 *Let A be a self-adjoint partial differential operator of Dirac type on a closed manifold M with* $\ker(A) = \{0\}$. *Then*

$$a_n^\eta(F,A) = \tfrac{1}{2}\mathrm{res}_{\,s=m-n}\Gamma(\tfrac{s+1}{2})\eta(s,F,A)\,.$$

We have assumed $\ker(P) = \{0\}$ and $\ker(A) = \{0\}$ to simplify the discussion. There are appropriate extensions of Lemmas 1.3.7 and 1.3.8 to the more general situation after adjusting for the effect of the kernels involved.

1.3.5 Index Theory

The invariants $a_n(x,\cdot)$ play a central role in index theory. Let

$$A : C^\infty(V_1) \to C^\infty(V_2)$$

be a partial differential operator over a closed Riemannian manifold M. We assume the bundles V_1 and V_2 have Hermitian structures. We say that this is an *elliptic complex* if $\sigma_L(A)(\xi) \in \mathrm{End}\,(V_1,V_2)$ is an isomorphism for all $\xi \neq 0$ or, equivalently, if the operators A^*A and AA^* are elliptic; the symbolic spectrum of these operators is then $(0,\infty)$. The *index* of A is given by setting

$$\begin{aligned}\mathrm{index}\,(A) :&= \dim\ker(A^*A) - \dim\ker(AA^*)\\ &= \dim\ker(A) - \dim\ker(A^*)\,.\end{aligned}$$

For example, let

$$d+\delta : C^\infty(\Lambda^{\mathrm{even}}(M)) \to C^\infty(\Lambda^{\mathrm{odd}}(M))$$

be the *de Rham complex* and let

$$\chi(M) := \sum_{p=0}^{m} (-1)^p H^p(M;\mathbb{R})$$

be the *Euler-Poincaré characteristic* of the manifold M. Then we may use Theorem 1.2.4 to see that

$$\operatorname{index}(d+\delta) = \chi(M)\,.$$

We define the *supertrace heat asymptotics* by

$$a_n(x,A) := a_n(x,A^*A) - a_n(x,AA^*)\,.$$

Theorem 1.3.9 *Let $A : C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over a closed m dimensional Riemannian manifold M. Then*

$$\int_M a_n(x,A)dx = \begin{cases} \operatorname{index}(A) & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m\,. \end{cases}$$

Proof: We use an observation due to Bott in the proof; this observation is the foundation of heat equation proofs of the index theorem. Let $D_1 := A^*A$ and $D_2 := AA^*$ be the associated self-adjoint elliptic operators on $C^\infty(V_1)$ and $C^\infty(V_2)$, respectively. Let

$$E_\lambda^i := \{\phi \in C^\infty(V_i) : D_i\phi = \lambda\phi\}\,.$$

Since $AD_1 = D_2A$, A intertwines these eigenspaces. Since $A^*A = \lambda$ on E_λ^1 and $AA^* = \lambda$ on E_λ^2,

$$A : E_\lambda^1 \to E_\lambda^2$$

is an isomorphism for $\lambda \neq 0$. Thus $\dim E_\lambda^1 = \dim E_\lambda^2$ for $\lambda \neq 0$. We have

$$\begin{aligned} & \operatorname{Tr}_{L^2}\left\{e^{-tD_1}\right\} - \operatorname{Tr}_{L^2}\left\{e^{-tD_2}\right\} \\ = \quad & \sum_\lambda e^{-t\lambda}\left\{\dim(E_\lambda^1) - \dim(E_\lambda^2)\right\} \\ = \quad & \dim(E_0^1) - \dim(E_0^2) = \operatorname{index}(A)\,. \end{aligned}$$

The Theorem now follows by equating terms in the associated asymptotic series; only the constant term can be non-zero. □

We can apply this result to the *Witten Laplacian* of Section 1.2.6 to get a local formula for the Euler-Poincaré characteristic χ of a closed Riemannian manifold.

Lemma 1.3.10 *Let (M,g) be a closed Riemannian manifold. Let ϕ be a smooth function on M which defines the Witten Laplacian Δ_ϕ^p. Define the local supertrace heat asymptotics by setting $a_{n,m}^{d+\delta}(\phi,g)(x) := \sum_p (-1)^p a_n(x,\Delta_\phi^p)$. Then*

$$\int_M a_{n,m}^{d+\delta}(\phi,g)(x)dx = \begin{cases} 0 & \text{if } n \neq m, \\ \chi(M) & \text{if } n = m\,. \end{cases}$$

Proof: Let

$$\iota(\phi,g) := \operatorname{index}(d+\delta)_\phi : C^\infty(\Lambda^{\text{even}}(M)) \to C^\infty(\Lambda^{\text{odd}}(M))\,.$$

By Theorem 1.3.9,

$$\int_M a_{n,m}^{d+\delta}(\phi,g)(x)dx = \begin{cases} \iota(\phi,g) & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m\,. \end{cases}$$

Since the left hand side is given by a local formula, we see $\iota(\phi,g)$ is a continuous integer valued invariant and hence constant under perturbations of ϕ and g. We may therefore set $\phi = 0$ and use Theorem 1.2.4 to see

$$\iota(\phi,g) = \iota(0,g) = \textstyle\sum_p (-1)^p \dim H^p(M;\mathbb{R}) = \chi(M). \qquad \square$$

Remark 1.3.11 We will generalize this result if ϕ satisfies Neumann boundary conditions to the context of manifolds with boundary subsequently in Lemma 1.5.10. This will lead to a heat equation proof of the classical Chern-Gauss-Bonnet theorem for manifolds with boundary in Theorem 1.9.2.

1.3.6 Heat content asymptotics

The *heat content asymptotics* provide another family of local invariants.

Theorem 1.3.12 *Let P be a d^{th} order partial differential operator on a closed Riemannian manifold with* $\operatorname{Spec}_\sigma(P) = (0,\infty)$. *If $\phi \in C^\infty(V)$ and if $\rho \in C^\infty(V^*)$, then we define*

$$\beta(\phi,\rho,P)(t) = \textstyle\int_M \langle e^{-tP}\phi,\rho\rangle dx\,.$$

1. *As $t \downarrow 0$, there is a complete asymptotic series*

$$\beta(\phi,\rho,P)(t) \sim \textstyle\sum_{n\geq 0} t^{n/d}\beta_n(\phi,\rho,P)\,.$$

2. *If $j = dk$, then $\beta_j(\phi,\rho,P) = (-1)^k\frac{1}{k!}\int_M\langle P^k\phi,\rho\rangle dx$.*
3. *If j is not divisible by d, then $\beta_j(\phi,\rho,P) = 0$.*

Proof: Assertion (1) can be established using the *Seeley calculus* [339]. Let $u := e^{-tP}\phi$. Then we have

$$\begin{aligned} &\sum_{n=0}^\infty \tfrac{n}{d} t^{(n-d)/d}\beta_n(\phi,\rho,P) \sim \partial_t\beta(\phi,\rho,P) = \int_M \langle\partial_t u,\rho\rangle dx \\ = \quad &-\int_M\langle Pu,\rho\rangle dx = -\int_M\langle u,\tilde P\rho\rangle dx \sim -\sum_{k=0}^\infty t^{k/d}\beta_k(\phi,\tilde P\rho,P)\,. \end{aligned}$$

We equate the coefficients of corresponding powers of t to obtain the recursion relation

$$\tfrac{n}{d}\beta_n(\phi,\rho,P) = -\beta_{n-d}(\phi,\tilde P\rho,P)\,. \qquad (1.3.e)$$

Since $u|_{t=0} = \phi$, $\beta_0(\phi,\rho) = \int_M\langle\phi,\rho\rangle dx$. We now use induction to see

$$\begin{aligned} \beta_{dk}(\phi,\rho,P) &= -\tfrac{1}{k}\beta_{dk-d}(\phi,\tilde P\rho,P) = \dots = (-1)^k\tfrac{1}{k!}\beta_0(\phi,\tilde P^k\rho,P) \\ &= (-1)^k\tfrac{1}{k!}\langle\phi,\tilde P^k\rho\rangle_{L^2} = (-1)^k\tfrac{1}{k!}\langle P^k\phi,\rho\rangle_{L^2}\,. \end{aligned}$$

Assertion (1) now follows. If n is **not** divisible by d, we argue similarly to see

$$\begin{aligned}\beta_n(\phi,\rho,P) &= -\tfrac{d}{n}\beta_{n-d}(\phi,\tilde{P}\rho,P)=...\\ &= (-1)^k\tfrac{d}{n}\tfrac{d}{n-d}\cdots\tfrac{d}{n-kd}\beta_{n-kd}(\phi,\tilde{P}^k\rho,P)\,.\end{aligned}$$

Assertion (2) now follows since $\beta_{n-kd}=0$ for $n-kd<0$. $\square$

1.4 Boundary ellipticity

If ∂M is non-empty, then we must impose suitable boundary conditions. We begin by introducing the classical *Lopatinskij-Shapiro* condition. We refer to Grubb [228] for further details concerning the material of this section.

Rather than working with the symbolic spectrum, it is convenient to work with a complementary subset. We suppose that $\mathcal{U}$ is a *conical* subset of $\mathbb{C}$. This means that if $\lambda\in\mathcal{U}$, then $t\lambda\in\mathcal{U}$ for any $t\geq 0$. Thus necessarily $0\in\mathcal{U}$.

If P is an operator of Laplace type, respectively of Dirac type, then we shall take $\mathcal{U}=\mathcal{C}$, respectively $\mathcal{U}=\mathcal{K}$, where

$$\mathcal{C}:=\mathbb{C}-(0,\infty)\quad\text{and}\quad\mathcal{K}:=\mathbb{C}-(0,\infty)-(-\infty,0)\,. \tag{1.4.a}$$

Let P be an elliptic operator of order $d>0$ on $C^\infty(V)$. Fix a connection ∇ on V; if P is an operator of Laplace type, then we shall take the connection of Lemma 1.2.1; if P is an operator of Dirac type, then we shall take a compatible connection. To simplify the notation, we set

$$W:=W_0\oplus...\oplus W_{d-1}\quad\text{where}\quad W_i:=V|_{\partial M}\,.$$

Let $\nabla^\nu_{e_m}\phi$ denote the ν^{th} covariant derivative of ϕ with respect to the inward unit geodesic normal vectorfield e_m. If $\phi\in C^\infty(V)$, then the *Cauchy data map* $\bar{\gamma}:C^\infty(V)\to C^\infty(W)$ is defined by setting

$$\bar{\gamma}\phi:=\phi|_{\partial M}\oplus(\nabla_{e_m}\phi)|_{\partial M}\oplus...\oplus(\nabla^{d-1}_{e_m}\phi)|_{\partial M}\in C^\infty(W)\,.$$

One can use a partition of unity to see that

Lemma 1.4.1 *The Cauchy data map $\bar{\gamma}:C^\infty(V)\to C^\infty(W)$ is surjective.*

Let $\mathcal{W}=\mathcal{W}_0\oplus...\oplus\mathcal{W}_{d-1}$ be an auxiliary vector bundle defined over ∂M; it is permissible to take $\mathcal{W}_i$ to be empty for certain values of i if $d\geq 2$. We assume that

$$\dim(\mathcal{W})=\tfrac{d}{2}\dim(V)\,.$$

We let

$$B:C^\infty(W)\to C^\infty(\mathcal{W})$$

be a smooth tangential partial differential operator which is defined on ∂M. We decompose $B=(B_{ji})$ for

$$B_{ji}:C^\infty(W_i)\to C^\infty(\mathcal{W}_j)\,.$$

This is a slight notational change from the convention employed in [189]. We shall regard a section to W_i as having order i since it will be thought of as arising from the i^{th} normal covariant derivative of a section to V. Thus we shall assume that

$$\operatorname{order}(B_{ji}) \leq j-i\,.$$

In particular, we shall set $B_{ji}=0$ for $j<i$.

Our *boundary operator* is then defined by setting $\mathcal{B} := B\circ\bar{\gamma}$, i.e.

$$\mathcal{B}\phi = \oplus_j\{B_{j0}(\phi|_{\partial M}) + B_{j1}(\nabla_{e_m}\phi|_{\partial M}) + \ldots + B_{jj}(\nabla_{e_m}^j\phi|_{\partial M})\}\,.$$

In the context of the singular structures defined in Section 1.2.8, we modify this definition slightly by replacing ∂M by the brane Σ and by replacing ϕ by (ϕ_+,ϕ_-). We suppress this additional technical fuss for the present in the interest of notational simplicity; we shall return to this point in Sections 1.6.1 and 1.6.3 when we discuss transmission and transfer boundary conditions.

Both W and $\mathcal{W}$ are examples of *graded vector bundles*. We take the grading into affect when considering the symbol. Let ζ be a cotangent vector on ∂M. We define the *graded leading symbol* of B by setting

$$\sigma_L^g(B_{ji})(y,\zeta) := \begin{cases} \sigma_L(B_{ji})(y,\zeta) & \text{if } \operatorname{order}(B_{ji}) = j-i, \\ 0 & \text{if } \operatorname{order}(B_{ji}) < j-i. \end{cases}$$

This is invariantly defined as a map

$$\sigma_L^g : T^*(\partial M) \to \operatorname{Hom}(W,\mathcal{W})\,.$$

We can now state the *Lopatinskij-Shapiro condition*. Let P be an elliptic d^{th} order partial differential operator. Let

$$0\in\mathcal{U}\subseteq \operatorname{Spec}_\sigma(P)^c$$

be a conical subset of $\mathbb{C}$ which is contained in the complement of the symbolic spectrum of P. Let $\vec{a}=(a_1,\ldots,a_{m-1})$ be a collection of non-negative integers. Let

$$\partial_y^{\vec{a}} := (\partial_1^y)^{a_1}\ldots(\partial_{m-1}^y)^{a_m}$$

denote multiple tangential partial differentiation and let ∂_r be the inward unit normal vector field. Express P in the form

$$P=\sum_{\vec{a},k} a_{\vec{a},k}\partial_y^{\vec{a}}\partial_r^k\,.$$

We formally replace $\partial_y^{\vec{a}}$ by $(\sqrt{-1})^{|\vec{a}|}\zeta^{\vec{a}}$ and suppress the lower order terms to define the following equations for a function $f=f(r)$:

$$\left\{\sum_{|\vec{a}|=d-k}(\sqrt{-1})^{|\vec{a}|}a_{\vec{a},k}\zeta^{\vec{a}}\partial_r^k-\lambda\right\}f(r)=0, \tag{1.4.b}$$

$$\lim_{r\to\infty} f(r)=0\,. \tag{1.4.c}$$

Equation (1.4.b) arises from taking a partial Fourier transform in the tangential variables only. Equation (1.4.c) is a growth condition.

Definition 1.4.2 We say that $(P, \mathcal{B})$ is *elliptic with respect to a cone* $\mathcal{U} \subset \mathbb{C}$ if the following two conditions are satisfied:

1. $\mathrm{Spec}_\sigma(P) \subset \mathcal{U}^c$.
2. For any $(0,0) \neq (\zeta, \lambda) \in T^*(\partial M) \times \mathcal{U}$ and for any $w \in \mathcal{W}$, there exists a unique solution to Equations (1.4.b) and (1.4.c) such that

$$\sigma_L^g(B)(y,\zeta)\bar{\gamma} f = w\,.$$

Theorem 1.3.2 generalizes to yield the following *elliptic regularity* result:

Theorem 1.4.3 *Let P be a d^{th} order partial differential operator on compact Riemannian manifold M with smooth boundary ∂M. If $(P, \mathcal{B})$ is elliptic with respect to the cone $\{0\}$ and if $P\phi = 0$ in the distributional sense on M, then ϕ is smooth on M.*

1.4.1 The heat equation

Let $(P, \mathcal{B})$ be elliptic with respect to the cone $\mathcal{C}$. Let $\phi \in C^\infty(V)$. There is a unique solution $u = u(x;t)$ to the equations

$$\begin{array}{lll} (\partial_t + P)u = 0 \quad \text{for} \quad t > 0 & \text{(evolution equation)} & \\ \mathcal{B}u = 0 \quad \text{for} \quad t > 0 & \text{(boundary condition)} & \qquad (1.4.\mathrm{d}) \\ u|_{t=0} = \phi & \text{(initial condition).} & \end{array}$$

We shall let the operator $\phi \to u$ be denoted by $e^{-tP_\mathcal{B}}$; this is the *fundamental solution of the heat equation.*

If $\phi \in L^2$, then $u(x;t)$ is smooth in $(x;t)$ for $t > 0$ so the *evolution equation* and *boundary condition* are to be understood in the usual sense. However, a bit of care is needed with the *initial condition.* Even if ϕ is smooth, it need not satisfy the boundary condition. We adopt the convention that the initial condition $u(x;t) = \phi(x)$ means that $\lim_{t\downarrow 0} u(\cdot;t) = \phi$ in the *distributional sense.* This means that

$$\lim_{t\downarrow 0} \int_M \langle u(x;t), \rho(x)\rangle dx = \int_M \langle \phi(x), \rho(x)\rangle dx \quad \text{for all} \quad \rho \in C^\infty(V^*)\,.$$

We adopt this convention henceforth in the interest of brevity and clarity whenever writing such an initial condition for a manifold with boundary.

It is worth digressing briefly to provide an example of this phenomena. We will return to this example subsequently in Section 2.3.

Example 1.4.4 Let $M = [0, \pi]$ be the interval, let $D = -\partial_x^2$, and let $\phi = 1$. We will show presently in Example 1.5.12 that the spectral resolution of the Laplacian with Neumann boundary conditions on the interval is given by

$$\left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \sin nx, n^2 \right\}_{n=1}^\infty\,.$$

Consequently, the associated *Fourier coefficients on the interval* are

$$\sigma_n(\phi) := \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \sin(nx)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Separation of variables now yields

$$u(x;t) = \sum_{n-\mathrm{odd}} e^{-tn^2} \frac{2\sqrt{2}}{n\sqrt{\pi}} \sin(nx)\,.$$

This series converges in the C^∞ topology for $t > 0$. If $x \in (0,\pi)$ and if $t = 0$, then the series converges conditionally, but not absolutely, to $\phi(x) = 1$. Finally, we note that

$$u(0;0) = u(\pi;0) = 0 \quad \text{while} \quad \phi(0) = \phi(\pi) = 1\,.$$

1.4.2 Heat trace asymptotics

Let $P_{\mathcal{B}}$ be the closure of the operator P on the space of smooth sections to V satisfying the boundary condition $\mathcal{B}\phi = 0$. Theorem 1.3.5 generalizes to this setting to give the *heat trace asymptotics.*

Theorem 1.4.5 *Let P be a d^{th} order partial differential operator on a compact Riemannian manifold M with smooth boundary ∂M. Assume that $(P, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$.*

1. *If $t > 0$, then the operator $e^{-tP_{\mathcal{B}}}$ is an infinitely smoothing operator which is of trace class on L^2. There exists a smooth kernel function $K(t,x,x_1,P,\mathcal{B})$ defining a linear map from V_{x_1} to V_x so that for $t > 0$,*

$$e^{-tP_{\mathcal{B}}}\phi(x;t) = \textstyle\int_M K(t,x,x_1,P,\mathcal{B})\phi(x_1)dx_1\,.$$

2. *Let $F \in C^\infty(\mathrm{End}\,(V))$ be an auxiliary smooth endomorphism. Then*

$$\mathrm{Tr}_{L^2}\{Fe^{-tP_{\mathcal{B}}}\} = \textstyle\int_M \mathrm{Tr}_{V_x}\{F(x)K(t,x,x,P,\mathcal{B})\}dx\,.$$

As $t \downarrow 0$, there is a complete asymptotic expansion of the form

$$\mathrm{Tr}_{L^2}\{Fe^{-tP_{\mathcal{B}}}\} \sim \textstyle\sum_{n\geq 0} t^{(n-m)/d} a_n(F,P,\mathcal{B})\,.$$

3. *Let the interior invariants $e_n(x,P)$ be as described in* Theorem 1.3.5. *Let ∇ be an auxiliary connection on V. There exist local endomorphism valued invariants $e_{n,k}$ defined on the boundary of M so that*

$$\begin{aligned} a_n(F,P,\mathcal{B}) \quad &= \quad \textstyle\int_M \mathrm{Tr}_{V_x}\{F(x)e_n(x,P)\}dx \\ &+ \quad \textstyle\sum_{k<n} \int_{\partial M} \mathrm{Tr}_{V_x}\{\nabla_{e_m}^k F(y) \cdot e_{n,k}(y,P,\mathcal{B})\}dy\,. \end{aligned}$$

As for the case of manifolds without boundary, it is convenient to introduce the corresponding scalar invariants. Let

$$a_{n,k}(y,P,\mathcal{B}) := \mathrm{Tr}_{V_x}\left\{e_{n,k}(y,P,\mathcal{B})\right\}. \tag{1.4.e}$$

We then have that if f is a scalar function, then

$$a_n(f\cdot\mathrm{Id}\,,P,\mathcal{B})=\int_M f(x)a_n(x,P)dx$$
$$+\sum_{k=0}^{n-1}\int_{\partial M}\nabla_{e_m}^k f(y)\cdot a_{n,k}(y,P,\mathcal{B})dy \quad \text{so}$$
$$\mathrm{Tr}_{L^2}\{e^{-tP_{\mathcal{B}}}\}\sim\sum_{n=0}^{\infty}t^{(n-m)/d}\left\{\int_M a_n(x,P)dx+\int_{\partial M}a_{n,0}(y,P,\mathcal{B})dy\right\}.$$

If $\mathcal{B}$ denotes spectral boundary conditions (see Section 1.6.6), then the situation is considerably more complicated. The asymptotic series here can contain non-local terms and also log terms. However, these terms do not appear until the constant term and thus play no role in our analysis. In the interest of simplicity, we omit a fuller discussion of this context and refer instead to Grubb [226, 227, 229] and to Grubb and Seeley [233, 234, 235]. We shall content ourselves with the following result which is sufficient to the task at hand giving the *heat trace asymptotics for spectral boundary conditions.*

Theorem 1.4.6 *Let $P:C^\infty(V_1)\to C^\infty(V_2)$ be an elliptic complex of Dirac type. Let $D=P^*P$ be the associated operator of Laplace type. Impose spectral boundary conditions $\mathcal{B}$. Let F be an auxiliary smooth endomorphism of V_1. Then there is an asymptotic series as $t\downarrow 0$ of the form*

$$\mathrm{Tr}_{L^2}(Fe^{-tD_{\mathcal{B}}})\sim\textstyle\sum_{0\le k\le m-1}a_k(F,D,\mathcal{B})t^{(k-m)/2}+O(t^{-1/8}).$$

1.4.3 Heat content asymptotics

The *heat content asymptotics* are not discussed explicitly by Greiner [224] or by Seeley [341]. However the calculations of the parametrix discussed there generalize immediately to establish the existence of the appropriate asymptotics for the heat content function; we refer to Grubb [228] for an excellent discussion of the heat equation in the context of elliptic boundary value problems. We also note that work of Kozlov [262] provides an alternative approach to the matter at hand.

Theorem 1.4.7 *Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M. Let P be a d^{th} order partial differential operator on M. Assume $(P,\mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$. Let $\phi\in C^\infty(V)$ and $\rho\in C^\infty(V^*)$.*

1. *As $t\downarrow 0$, there exists a complete asymptotic expansion of the form*

$$\beta(\phi,\rho,P,\mathcal{B})(t):=\textstyle\int_M\langle e^{-tP_{\mathcal{B}}}\phi,\rho\rangle(x,t)dx\sim\sum_{n\ge 0}t^{n/d}\beta_n(\phi,\rho,P,\mathcal{B}).$$

2. *There exist local invariants β_n^M and $\beta_n^{\partial M}$ which are bilinear in the jets of ϕ and of ρ so that*

$$\beta_n(\phi,\rho,P,\mathcal{B}) \;=\; \textstyle\int_M\beta_n^M(\phi,\rho,P)(x)dx+\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,P,\mathcal{B})(y)dy.$$

The interior invariants β_n^M are described by Assertions (2) and (3) of Theorem 1.3.12 as we can take

$$\beta_n^M(\phi,\rho,P) = \begin{cases} 0 & \text{if } \ d \text{ does not divide } n, \\ (-1)^k \frac{1}{k!}\langle P^k\phi,\rho\rangle & \text{if } \ n = dk\,. \end{cases} \tag{1.4.f}$$

The interior invariants are not unique. Thus by using an appropriate Green's formula we could also take

$$\beta_{dk}^M(\phi,\rho,P) = (-1)^k \tfrac{1}{k!}\langle \phi, \tilde P^k \rho\rangle$$

at the cost of changing the boundary integrand $\beta_{dk}^{\partial M}$ appropriately.

1.4.4 Operators of Laplace type

Let $\mathcal{B}$ define a boundary condition for an operator D of Laplace type. Then

$$\begin{aligned} B &= \begin{pmatrix} b_{00} & 0 \\ b_{10} + b_{10}^a \nabla_{e_a} & b_{11} \end{pmatrix}, \\ \sigma_L^g(B)(\zeta) &= \begin{pmatrix} b_{00} & 0 \\ \sqrt{-1} b_{10}^a \zeta_a & b_{11} \end{pmatrix}, \\ \mathcal{B}\phi &= \begin{pmatrix} b_{00}\phi \\ b_{10}\phi + b_{10}^a \nabla_{e_a}\phi + b_{11}\nabla_{e_m}\phi \end{pmatrix}\Bigg|_{\partial M}. \end{aligned}$$

If P is an operator of Laplace type, then Equation (1.4.b) becomes

$$(-\partial_r^2 + |\zeta|^2 - \lambda) f(r) = 0\,. \tag{1.4.g}$$

Since $0 \neq (\zeta,\lambda) \in T^*\partial M \times \mathcal{C}$,

$$|\zeta|^2 - \lambda \notin (-\infty, 0]\,.$$

Choose the branch of the square root function so that

$$\Re\sqrt{|\zeta|^2-\lambda} > 0\,.$$

The solutions to Equation (1.4.g) are exponentials of the form

$$f(r) = e^{-r\sqrt{|\zeta|^2-\lambda}}\phi_- + e^{r\sqrt{|\zeta|^2-\lambda}}\phi_+\,.$$

The solutions decaying as $r \to \infty$ have $\phi_+ = 0$ so $f(r) = e^{-r\sqrt{|\zeta|^2-\lambda}}\phi_-$. Thus

$$\bar\gamma f = \phi_- \oplus -\sqrt{|\zeta|^2-\lambda}\,\phi_-\,.$$

The following operator will play a crucial role in our subsequent analysis. It will be used to express the *Lopatinskij-Shapiro condition* on the symbolic level when considering boundary conditions for operators of Laplace type. Set

$$\mathfrak{b}(\zeta,\lambda) : \phi \to \begin{pmatrix} b_{00}\phi \\ \sqrt{-1} b_{10}^a \zeta_a \phi - b_{11}\sqrt{|\zeta|^2-\lambda}\,\phi \end{pmatrix} \tag{1.4.h}$$

The following Lemma is now immediate:

Lemma 1.4.8 *Let D be an operator of Laplace type. Let $\mathcal{B}$ be as above. Then $(D,\mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$ if and only if $\mathfrak{b}(\zeta,\lambda)$ is an isomorphism from $V|_{\partial M}$ to $\mathcal{W}$ for any $(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{C}$.*

1.4.5 Operators of Dirac type

Let $d=1$ and let $P=\gamma_i\nabla_{e_i}+\psi_P$ be an operator of Dirac type. By Lemma 1.3.1, P is elliptic with respect to the cone

$$\mathcal{K}:=\mathbb{C}-(-\infty,0)-(0,\infty)\,.$$

A boundary condition is then an endomorphism

$$B:V|_{\partial M}\to\mathcal{W}\text{ where }\dim\mathcal{W}=\tfrac{1}{2}\dim V\,.$$

If $(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{K}$, then

$$|\zeta|^2-\lambda^2\notin(-\infty,0]\,.$$

We choose the branch of the square root function so

$$\Re(\sqrt{|\zeta|^2-\lambda^2})>0\,.$$

We will use the following operator in our discussion of the *Lopatinskij-Shapiro condition.* Set

$$\Xi(\zeta,\lambda):=\sqrt{-1}\gamma_m\gamma_a\zeta_a-\gamma_m\lambda\,. \tag{1.4.i}$$

The Clifford commutation rules show

$$\Xi(\zeta,\lambda)^2=(|\zeta|^2-\lambda^2)\mathrm{Id}\,. \tag{1.4.j}$$

Let $V_\pm(\zeta,\lambda)$ be the associated eigenspaces. They are defined by

$$V_\pm(\zeta,\lambda):=\{v\in V|_{\partial M}:\Xi(\zeta,\lambda)v=\pm\sqrt{|\zeta|^2-\lambda^2}v\}\,. \tag{1.4.k}$$

Lemma 1.4.9 *Let P be an operator of Dirac type and let B be a linear map from $V|_{\partial M}$ to $\mathcal{W}$. Then (P,B) is elliptic with respect to the cone $\mathcal{K}$ if and only if $B:V_\pm(\zeta,\lambda)\xrightarrow{\approx}\mathcal{W}$ for all $(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{K}$.*

Proof: Equation (1.4.b) takes the form

$$(\gamma_m\partial_r+\sqrt{-1}\gamma_a\zeta_a-\lambda)f(r)=0\,.$$

As $\gamma_m^2=-\mathrm{Id}$, we multiply by $-\gamma_m$ to see equivalently that

$$(\partial_r-\Xi(\zeta,\lambda))f(r)=0\,. \tag{1.4.l}$$

We use Equations (1.4.j) and (1.4.k). Decompose $f(0)=v_++v_-$ where we have $v_\pm\in V_\pm$. The solutions to Equation (1.4.l) are then given by

$$f(r)=e^{r\sqrt{|\zeta|^2-\lambda^2}}v_++e^{-r\sqrt{|\zeta|^2-\lambda^2}}v_-\ \text{ for }v_\pm\in V_\pm\,.$$

Since $e^{r\sqrt{|\zeta|^2-\lambda^2}}v_+$ increases exponentially as $r\to\infty$, we must have $v_+=0$. Thus $(P,\mathcal{B})$ is elliptic with respect to $\mathcal{K}$ if and only if

$$B:V_-\to\mathcal{W}$$

is an isomorphism for

$$(0,0) \neq (\zeta, \lambda) \in T^*(\partial M) \times \mathcal{K} .$$

The same condition holds for V_+ since $V_+(\zeta,\lambda) = V_-(-\zeta,-\lambda)$. $\square$

In discussing spectral boundary conditions in Section 1.6.6, we will permit B to be a 0^{th} order pseudo-differential operator, but we suppress this technical complication for the moment in the interest of notational simplicity. At that point, the relevant sign conventions will be crucial.

1.4.6 Induced second order boundary conditions

Let $P = \gamma_i \nabla_{e_i} + \psi_P$ be an operator of Dirac type. Let $D = P^2$ be the associated operator of Laplace type. Let $\mathcal{B}_1$ define a 0^{th} order boundary condition; let $\mathcal{B}_2 := \mathcal{B}_1 \oplus \mathcal{B}_1 P$ be the *induced boundary operator* for D. Let $\zeta \in T^*\partial M$ and let $\lambda, \mu \in \mathbb{C}$. Adopt the notation of Equations (1.4.h) and (1.4.i). Let

$$\begin{aligned}
&\Xi(\zeta,\lambda) := \sqrt{-1}\gamma_m \gamma_a \zeta_a - \gamma_m \lambda, \\
&\mu := \sqrt{|\zeta|^2 - \lambda^2}, \\
&\sigma(\zeta,\mu) := \sqrt{-1}\gamma_a \zeta_a - \gamma_m \mu, \quad \text{and} \\
&\mathfrak{b}_2(\zeta,\lambda^2) := \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_1 \sigma(\zeta,\mu) \end{pmatrix} .
\end{aligned}$$

We use the Clifford commutation relations to compute that

$$\Xi(\zeta,\lambda)^2 = (|\zeta|^2 - \lambda^2)\mathrm{Id} \quad \text{and} \quad \sigma(\zeta,\mu)^2 = (|\zeta|^2 - \mu^2)\mathrm{Id} .$$

Lemma 1.4.10 *Let* $(0,0) \neq (\zeta,\lambda) \in T^*(\partial M) \times \mathcal{K}$*. Let* $\mu = \sqrt{|\zeta|^2 - \lambda^2}$.

1. $\Xi(\zeta,\lambda)v = -\mu v$ *if and only if* $\sigma(\zeta,\mu)v = \lambda v$.
2. *Let* $\lambda = 0$ *and* $\mu = |\zeta|$*. Then* $\ker\{\sigma(\zeta,\mu)\} = V_-(\zeta,0)$ *and* $\sigma(\zeta,\mu)$ *is an isomorphism from* $V_+(\zeta,0)$ *onto* $V_-(\zeta,0)$.

Proof: Assertion (1) follows from the following chain of equivalent equations

$$\begin{aligned}
& \Xi(\zeta,\lambda)v = -\mu v && \Leftrightarrow \quad (\sqrt{-1}\gamma_m\gamma_a\zeta_a - \gamma_m\lambda)v = -\mu v \\
\Leftrightarrow \quad & (\sqrt{-1}\gamma_a\zeta_a - \lambda)v = \gamma_m \mu v && \Leftrightarrow \quad (\sqrt{-1}\gamma_a\zeta_a - \gamma_m\mu)v = \lambda v \\
\Leftrightarrow \quad & \sigma(\zeta,\mu)v = \lambda v .
\end{aligned}$$

If $\lambda = 0$, then $\ker \sigma(\zeta,\mu) = V_-(\zeta,0)$ by Assertion (1). As $\dim V_+ = \dim V_-$ and as $\sigma(\zeta,\mu)^2 = 0$, Assertion (2) now follows. $\square$

We have the following result *relating ellipticity conditions*:

Theorem 1.4.11 *Let* P *be an operator of Dirac type. Let* $\mathcal{B}_1 : V|_{\partial M} \to \mathcal{W}$ *define a* 0^{th} *order boundary condition for* P *where* $\dim \mathcal{W} = \frac{1}{2}\dim V$*. Let* $D := P^2$ *be the associated operator of Laplace type and let*

$$\mathcal{B}_2\phi := \mathcal{B}_1\phi \oplus \mathcal{B}_1 P\phi$$

be the associated boundary condition for D*. Then the following assertions are equivalent:*

1. $(P,\mathcal{B}_1)$ *is elliptic with respect to the cone* $\mathcal{K}$*.*

2. $(D,\mathcal{B}_2)$ *is elliptic with respect to the cone* $\mathcal{C}$*.*

Proof: We apply Lemmas 1.4.8, 1.4.9 and 1.4.10. Let $P=\gamma_i\nabla_{e_i}+\psi_P$ be an operator of Dirac type and let $D=P^2$ be the associated operator of Laplace type. If $(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{K}$, then

$$\mathfrak{b}_2(\zeta,\lambda^2)=\left(\begin{array}{c}\mathcal{B}_1\\ \mathcal{B}_1\sigma(\zeta,\mu)\end{array}\right)\quad\text{for}\quad \mu=\sqrt{|\zeta|^2-\lambda^2}\,.$$

To prove that Assertion (1) implies Assertion (2), we suppose that $(P,\mathcal{B}_1)$ is elliptic with respect to the cone $\mathcal{K}$ but that $(D,\mathcal{B}_2)$ is not elliptic with respect to the cone $\mathcal{C}$ and argue for a contradiction. Choose

$$(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{K}\quad\text{and}\quad 0\neq v$$

so

$$\mathfrak{b}_2(\zeta,\lambda^2)v=0\,.$$

Suppose that $\lambda\neq0$. Decompose

$$v=v_++v_-\quad\text{for}\quad\sigma(\zeta,\mu)v_\pm=\pm\lambda v_\pm\,.$$

Since $\mathfrak{b}(\zeta,\lambda^2)v=0$, we have that

$$\mathcal{B}_1(v_++v_-)=0\quad\text{and}\quad\mathcal{B}_1(\lambda v_+-\lambda v_-)=0\,.$$

Because $\lambda\neq0$, this implies $\mathcal{B}_1v_+=0$ and $\mathcal{B}_1v_-=0$ separately. One has

$$\begin{aligned}\sigma(\zeta,\mu)v_+&=\ \ \lambda v_+\ \Rightarrow\ \Xi(\zeta,\lambda)v_+\ \ =-\mu v_+\ \Rightarrow\ v_+\in V_-(\zeta,\lambda),\\ \sigma(\zeta,\mu)v_-&=-\lambda v_-\ \Rightarrow\ \Xi(\zeta,-\lambda)v_-=-\mu v_-\ \Rightarrow\ v_-\in V_-(\zeta,-\lambda)\,.\end{aligned}$$

Since $\mathcal{B}_1v_+=0$, since $\mathcal{B}_1v_-=0$, and since $\mathcal{B}_1$ is elliptic with respect to the cone $\mathcal{K}$, we have $v_+=0$ and $v_-=0$ so $v=0$ which is false.

Suppose that $\lambda=0$ so $\mu=|\zeta|$. By Lemma 1.4.10, $\ker\sigma(\zeta,\mu)=V_-(\zeta,0)$. This controls the Jordan normal form of $\sigma(\zeta,\mu)$. Decompose $v=v_++v_-$ where $\Xi(\zeta,0)v_\pm=\pm\mu v_\pm$. As $\sigma(\zeta,\mu)v_-=0$, $\mathcal{B}_1\sigma(\zeta,\mu)v_+=0$. Since

$$\sigma(\zeta,\mu)v_+\in\ker\sigma(\zeta,\mu)=V_-(\zeta,0)$$

$\sigma(\zeta,\mu)v_+=0$ as $(P,\mathcal{B}_1)$ is elliptic with respect to $\mathcal{K}$. Consequently $v_+=0$ so $v=v_-$. Since $\mathcal{B}_1v_-=0$, $v_-=0$ and thus $v=0$. This contradiction shows Assertion (1) implies Assertion (2).

Conversely, suppose that $(D,\mathcal{B}_2)$ is elliptic with respect to the cone $\mathcal{C}$. Let $(0,0)\neq(\zeta,\lambda)\in T^*(\partial M)\times\mathcal{K}$. Suppose $v_-\in V_-(\zeta,\lambda)\cap\ker(\mathcal{B}_1)$. By Lemma 1.4.10, we have $\sigma(\zeta,\mu)=\lambda v_-$. Thus $\mathfrak{b}_2v_-=0$ so $v_-=0$. □

1.4.7 The dual operator and the dual boundary condition

Recall that the dual operator $\tilde P$ on V^* is defined by the identity

$$\langle P\phi,\rho\rangle_{L^2}=\langle\phi,\tilde P\rho\rangle_{L^2}\quad\text{for all}\quad \phi\in C_0^\infty(V),\ \rho\in C_0^\infty(V^*)\,.$$

If D is an operator of Laplace type, then we use Lemma 1.2.1 to express $D = D(\nabla, E)$. We then have $\tilde{D} = D(\tilde{\nabla}, \tilde{E})$ where, by Lemma 1.2.2, $\tilde{\nabla}$ is the dual connection on V^* and $\tilde{E}$ is the dual endomorphism on V^*.

Definition 1.4.12 We say that an operator $\tilde{\mathcal{B}}$ on V^* defines the *dual or adjoint boundary condition* if the following properties are satisfied:

1. If $\phi \in C^\infty(V)$, if $\rho \in C^\infty(V^*)$, if $\mathcal{B}\phi = 0$, and if $\tilde{\mathcal{B}}\rho = 0$, then
$$\langle P\phi, \rho\rangle_{L^2} = \langle \phi, \tilde{P}\rho\rangle_{L^2} \,.$$
2. Let $\rho \in C^\infty(V^*)$. Suppose that $\langle P\phi, \rho\rangle_{L^2} = \langle \phi, \tilde{P}\rho\rangle_{L^2}$ for every $\phi \in C^\infty(V)$ with $\mathcal{B}\phi = 0$. Then $\tilde{\mathcal{B}}\rho = 0$.

Definition 1.4.13 Let P be a d^{th} order partial differential operator. Let $\mathcal{B}$ be a boundary condition so that $(P, \mathcal{B})$ is elliptic with respect to either the cone $\mathcal{C}$ or the cone $\mathcal{K}$. Let $\tilde{P}$ be the associated operator on V^* and let $\tilde{\mathcal{B}}$ define the adjoint boundary condition. Let V have a Hermitian innerproduct. We say that $(P, \mathcal{B})$ is *self-adjoint* if $P = \tilde{P}$ and if $\mathcal{B} = \tilde{\mathcal{B}}$ under the natural conjugate linear identification of V with V^*.

1.4.8 Green's formula

We shall need the *Green's formula* in our discussion of the dual operator subsequently. Although the proofs are elementary, the formulae are central to the subject. We begin by establishing a basic identity. Let $\omega = \omega_i e_i$ be a 1 form. Note that
$$\omega_{i;i} = e_i(\omega_i) - \Gamma_{iij}\omega_j \,. \tag{1.4.m}$$
We use *Stoke's theorem* and Lemma 1.2.7 to see:
$$\begin{aligned}\int_M \omega_{i;i} dx &= -\int_M (\delta\omega) dx \\ &= (-1)^m \int_M (\star_m d \star_1 \omega) dx = (-1)^m \int_M d \star_1 \omega \\ &= (-1)^m \int_{\partial M} \star_1 \omega = -\int_{\partial M} \omega_m dy \,.\end{aligned} \tag{1.4.n}$$
The proper signs involved are always a bit tricky as there are several different normalizations involved. The proper sign can be determined by making a computation on the half space in $\mathbb{R}^n$; we always use the inward unit normal.

We can now derive the *Green's formula for an operator of Dirac type*:

Lemma 1.4.14 *Let M be a compact Riemannian manifold. Let γ be a Clifford module structure on a vector bundle V over M. Use Lemma 1.1.7 to choose a compatible connection ∇. Let $P = \gamma_i \nabla_{e_i} + \psi_P$ be an operator of Dirac type on V. Set $\tilde{P} := -\tilde{\gamma}_i \tilde{\nabla}_{e_i} + \tilde{\psi}_P$ on $C^\infty(V^*)$. Then*
$$\textstyle\int_M \{\langle P\phi, \rho\rangle - \langle \phi, \tilde{P}\rho\rangle\} dx = -\int_{\partial M} \langle \gamma_m \phi, \rho\rangle dy \,.$$

Proof: Let $\phi \in C^\infty(V)$ and $\rho \in C^\infty(V^*)$. Let $\omega := \langle \gamma_i \phi, \rho \rangle e_i \in C^\infty(T^*M)$. Since $\nabla\gamma = 0$,

$$0 = \nabla_{e_i}\gamma_j - \gamma_j \nabla_{e_i} - \Gamma_{ijk}\gamma_k\,.$$

We use Equations (1.4.m) and (1.4.n) to compute

$$\begin{aligned}
- \quad & \int_{\partial M} \langle \gamma_m \phi, \rho \rangle dy = \int_M \omega_{i;i} dx = \int_M \Big\{ e_i \langle \gamma_i \phi, \rho \rangle - \Gamma_{iij}\omega_j \Big\} dx \\
= \quad & \int_M \Big\{ \langle \nabla_{e_i} \gamma_i \phi, \rho \rangle + \langle \gamma_i \phi, \tilde\nabla_{e_i} \rho \rangle - \Gamma_{iij} \langle \gamma_j \phi, \rho \rangle \Big\} dx \\
= \quad & \int_M \Big\{ \langle \gamma_i \nabla_{e_i} \phi, \rho \rangle + \langle \phi, \tilde\gamma_i \tilde\nabla_{e_i} \rho \rangle \Big\} dx \\
= \quad & \int_M \Big\{ \langle P\phi, \rho \rangle - \langle \phi, \tilde P \rho \rangle \Big\} dx\,. \qquad \square
\end{aligned}$$

It will be useful to have a similar Green's formula on the boundary for use in studying spectral boundary conditions in Section 1.6.6. Recall that we defined the *tangential Clifford module structure* in Equation (1.1.l) by setting

$$\gamma_a^T := -\gamma_m \gamma_a\,.$$

Lemma 1.4.15 *Let M be a compact Riemannian manifold. Let γ be a Clifford module structure on a vector bundle V over M and let ∇ be a compatible connection. Let $A := -\gamma_m \gamma_a \nabla_{e_a} + \psi_A$ be a operator of Dirac type on $V|_{\partial M}$. Set $\tilde A := -\tilde\gamma_m \tilde\gamma_a \tilde\nabla_{e_a} + \tilde\psi_A$ on $V^*|_{\partial M}$. Then*

$$\textstyle\int_{\partial M} \{ \langle A\phi, \rho \rangle - \langle \phi, \tilde A \rho \rangle \} dy = 0\,.$$

Proof: Lemma 1.4.14 may not be applied directly as $\nabla^{\partial M}\gamma^T$ is in general non-zero; see Equation (1.1.m). Set $\omega := \langle \gamma_m \gamma_a \phi, \rho \rangle e_a \in C^\infty(T^*\partial M)$. We apply Equation (1.4.n), replacing M by ∂M; the boundary correction term vanishes as ∂M is closed. Note that $L_{ac}\gamma_a\gamma_c = -L_{aa}$. We use Equation (1.4.m) to see

$$\begin{aligned}
0 \quad = \quad & \int_{\partial M} \langle -\gamma_m \gamma_a \phi, \rho \rangle_{:a} dy \\
= \quad & \int_{\partial M} \Big\{ e_a \langle -\gamma_m \gamma_a \phi, \rho \rangle + \Gamma_{aac} \langle \gamma_m \gamma_c \phi, \rho \rangle \Big\} dy \\
= \quad & \int_{\partial M} \Big\{ \langle -\nabla_{e_a} \gamma_m \gamma_a \phi, \rho \rangle + \langle -\gamma_m \gamma_a \phi, \tilde\nabla_{e_a} \rho \rangle + \langle \Gamma_{aac} \gamma_m \gamma_c \phi, \rho \rangle \Big\} dy \\
= \quad & \int_{\partial M} \Big\{ \langle -\gamma_m \gamma_a \nabla_{e_a} \phi, \rho \rangle + \langle -\Gamma_{amc} \gamma_c \gamma_a \phi, \rho \rangle + \langle -\Gamma_{aac} \gamma_m \gamma_c \phi, \rho \rangle \\
& \quad - \langle \Gamma_{aam} \gamma_m \gamma_m \phi, \rho \rangle + \langle \Gamma_{aac} \gamma_m \gamma_c \phi, \rho \rangle + \langle \phi, \tilde\gamma_m \tilde\gamma_a \tilde\nabla_{e_a} \rho \rangle \Big\} dy \\
= \quad & \int_{\partial M} \Big\{ \langle A\phi, \rho \rangle + L_{ac} \langle \gamma_c \gamma_a \phi, \rho \rangle + L_{aa} \langle \phi, \rho \rangle - \langle \phi, \tilde A \rho \rangle \Big\} dy \\
= \quad & \int_{\partial M} \Big\{ \langle A\phi, \rho \rangle - \langle \phi, \tilde A \rho \rangle \Big\} dy\,. \qquad \square
\end{aligned}$$

The Green's formula for exterior differentiation d and the formal adjoint, interior differentiation, δ, is derived similarly.

Lemma 1.4.16 *Let ϕ and ρ be smooth p forms on a compact Riemannian manifold. Then*

$$\textstyle\int_M\{(d\phi,\rho)-(\phi,\delta\rho)\}dx=-\int_{\partial M}(\phi,\mathfrak{i}(e_m)\rho)dy\,.$$

Proof: Let $\{e_i\}$ be a local orthonormal frame. Let $\mathfrak{e}_i:=\mathfrak{e}(e_i)$ and $\mathfrak{i}_i:=\mathfrak{i}(e_i)$ denote left exterior and left interior multiplication by e_i, respectively. Let ∇ be the Levi-Civita connection. By Lemma 1.2.5, $d=\mathfrak{e}_i\nabla_{e_i}$ and $\delta=-\mathfrak{i}_i\nabla_{e_i}$. Since $\nabla\mathfrak{e}=0$ and $\nabla\mathfrak{i}=0$,

$$\nabla_{e_i}\mathfrak{e}_j-\mathfrak{e}_j\nabla_{e_i}-\Gamma_{ijk}\mathfrak{e}_k=0\quad\text{and}\quad\nabla_{e_i}\mathfrak{i}_j-\mathfrak{i}_j\nabla_{e_i}-\Gamma_{ijk}\mathfrak{i}_k=0\,.$$

Let $\omega:=(\mathfrak{e}_i\phi,\rho)e_i\in C^\infty(T^*M)$. Equations (1.4.m) and (1.4.n) yield

$$\begin{aligned}&-\int_{\partial M}(\mathfrak{e}_m\phi,\rho)dy=\int_M\Big\{e_i(\mathfrak{e}_i\phi,\rho)-\Gamma_{iij}(\mathfrak{e}_j\phi,\rho)\Big\}dx\\
=\;&\int_M\Big\{((\nabla_{e_i}\mathfrak{e}_i-\Gamma_{iij}\mathfrak{e}_j)\phi,\rho)+(\mathfrak{e}_i\phi,\nabla_{e_i}\rho)\Big\}dx\\
=\;&\int_M\Big\{(\mathfrak{e}_i\nabla_{e_i}\phi,\rho)+(\phi,\mathfrak{i}_i\nabla_{e_i}\rho)\Big\}dx=\int_M\Big\{(d\phi,\rho)-(\phi,\delta\rho)\Big\}dx\,.\qquad\Box\end{aligned}$$

There is a similar formula for operators of Laplace type:

Lemma 1.4.17 *Let $D=D(\nabla,E)$ be an operator of Laplace type on a smooth vector bundle V over a compact Riemannian manifold M. Let $\tilde D=D(\tilde\nabla,\tilde E)$ be the corresponding operator of Laplace type on the dual bundle on V^*. Then*

$$\textstyle\int_M\{\langle D\phi,\rho\rangle-\langle\phi,\tilde D\rho\rangle\}dx=\int_{\partial M}\{\langle\phi_{;m},\rho\rangle-\langle\phi,\rho_{;m}\rangle\}dy\,.$$

Proof: We compute

$$\begin{aligned}&\int_M\Big\{\langle D\phi,\rho\rangle-\langle\phi,\tilde D\rho\rangle\Big\}dx\\
=\;&\int_M\Big\{-\langle(\nabla_{e_i}\nabla_{e_i}\phi-\Gamma_{iij}\nabla_{e_j}+E)\phi,\rho\rangle\\
&\qquad+\langle\phi,(\tilde\nabla_{e_i}\tilde\nabla_{e_i}-\Gamma_{iij}\tilde\nabla_{e_j}+\tilde E)\rho\rangle\Big\}dx\\
=\;&\int_M\Big\{-(\langle\nabla_{e_i}\phi,\rho\rangle_{;i}-\langle\nabla_{e_i}\phi,\tilde\nabla_{e_i}\rho\rangle)\\
&\qquad+(\langle\phi,\tilde\nabla_{e_i}\rho\rangle_{;i}-\langle\nabla_{e_i}\phi,\tilde\nabla_{e_i}\rho\rangle)\Big\}dx\\
=\;&\int_M\Big\{-\langle\nabla_{e_i}\phi,\rho\rangle_{;i}+\langle\phi,\tilde\nabla_{e_i}\rho\rangle_{;i}\Big\}dx\\
=\;&\int_{\partial M}\Big\{\langle\nabla_{e_m}\phi,\rho\rangle-\langle\phi,\tilde\nabla_{e_m}\rho\rangle\Big\}dy\,.\qquad\Box\end{aligned}$$

1.4.9 Spectral theory of self-adjoint operators

The following result generalizes Theorem 1.3.3 to this setting; it follows from standard elliptic operator theory, see, for example, Grubb [228].

Theorem 1.4.18 *Let P be a partial differential operator on a bundle V over a compact Riemannian manifold M. Let $\mathcal{B}$ be a local boundary condition. Assume that $(P, \mathcal{B})$ is elliptic with respect to the cone $\{0\}$ and that $P_{\mathcal{B}}$ is self-adjoint. Then:*

1. *There exists a complete orthonormal basis $\{\phi_i\}$ for $L^2(V)$ so that $\phi_i \in C^\infty(V)$, so that $P\phi_i = \lambda_i\phi_i$, and so that $\mathcal{B}\phi_i = 0$.*
2. *Order the eigenvalues so $|\lambda_1| \le |\lambda_2| \ldots$. There exists $\epsilon > 0$ and $i_0 > 0$ so $|\lambda_i| \ge i^\epsilon$ for $i \ge i_0$.*
3. *If $(P, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$, then only a finite number of the eigenvalues λ_i can be negative.*
4. *For every $k \ge 0$, there exists a constant C_k and an integer n_k so that $||\phi_i||_{\infty,k} \le C_k|\lambda_i|^{n_k}$ for $i \ge i_0$.*

1.5 Boundary conditions I

The choice of proper boundary conditions is crucial to our study so we present a brief introduction here to the boundary conditions we shall employ. Throughout this and the subsequent section, let D be an operator of Laplace type and let ∇ be the associated covariant derivative. If $\mathcal{B}$ is a boundary operator, then the associated boundary condition is defined by $\mathcal{B}\phi = 0$. The *realization* of D with respect to the boundary condition $\mathcal{B}$, which we shall denote by $D_{\mathcal{B}}$, is D acting on the closure of the set of smooth functions ϕ which satisfy this boundary condition, in other words, for those ϕ which satisfy $\mathcal{B}\phi = 0$. We adopt the notation of Equation (1.4.a) and let

$$\mathcal{C} := \mathbb{C} - (0, \infty) \quad \text{and} \quad \mathcal{K} := \mathbb{C} - (0, \infty) - (-\infty, 0)\,.$$

Let $\phi_{;m} = \nabla_{e_m}\phi$ denote the covariant derivative of ϕ with respect to the inward unit normal along the boundary.

1.5.1 Dirichlet boundary conditions

The Dirichlet boundary operators on V and V^* are defined by setting

$$\mathcal{B}\phi := \phi|_{\partial M} \quad \text{and} \quad \tilde{\mathcal{B}}\rho := \rho|_{\partial M}\,. \tag{1.5.a}$$

Solving the heat equation with Dirichlet boundary conditions corresponds physically to keeping the boundary immersed in ice water, in other words at 0 degrees temperature (Centigrade).

Lemma 1.5.1 *Let D be an operator of Laplace type. Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be the Dirichlet boundary operators defined in Equation* (1.5.a). *Then:*

1. *$(D, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$.*
2. *$\tilde{\mathcal{B}}$ defines the adjoint boundary operator for $\tilde{D}$ on $C^\infty(V^*)$.*

Proof: Since we are taking Dirichlet boundary conditions, the map $\mathfrak{b}(\zeta, \lambda)$ of Equation (1.4.h) is the identity map on $V|_{\partial M}$. The first assertion now follows by Lemma 1.4.8.

To prove the second assertion, we use Lemma 1.4.17 to see that

$$\langle D\phi, \rho\rangle_{L^2} - \langle \phi, \tilde{D}\rho\rangle_{L^2} = \int_{\partial M} \left\{ \langle \phi_{;m}, \tilde{\mathcal{B}}\rho\rangle - \langle \mathcal{B}\phi, \rho_{;m}\rangle \right\} dy\,.$$

This vanishes if $\mathcal{B}\phi = 0$ and if $\tilde{\mathcal{B}}\rho = 0$ so the first assertion of Definition 1.4.12 is satisfied. Conversely, if we have that $\langle D\phi, \rho\rangle_{L^2} - \langle \phi, \tilde{D}\rho\rangle_{L^2} = 0$ for all ϕ with $\phi|_{\partial M} = 0$, we then have

$$\int_{\partial M} \langle \phi_{;m}, \tilde{\mathcal{B}}\rho\rangle dy = 0\,. \tag{1.5.b}$$

We use Lemma 1.4.1 to see that we can choose $\phi \in C^\infty(V)$ so $\phi|_{\partial M} = 0$ and so $\phi_{;m}|_{\partial M}$ is arbitrary. Thus Equation (1.5.b) implies $\tilde{\mathcal{B}}\rho = 0$ so $\tilde{\mathcal{B}}$ defines the adjoint boundary condition. □

1.5.2 Neumann and Robin boundary conditions

The *Neumann boundary operator* is defined by

$$\mathcal{B}\phi = \phi_{;m}|_{\partial M}\,.$$

Solving the heat equation with Neumann boundary conditions corresponds physically to insulating the boundary.

More generally, suppose given an auxiliary endomorphism S of $V|_{\partial M}$. Let

$$\mathcal{B}\phi := (\phi_{;m} + S\phi)|_{\partial M} \quad \text{and} \quad \tilde{\mathcal{B}}\rho := (\rho_{;m} + \tilde{S}\rho)|_{\partial M} \tag{1.5.c}$$

define the *Robin boundary operators* on V and on V^*. Solving the heat equation with Robin boundary conditions corresponds to ensuring that the heat flow across the boundary is proportional to the temperature on the boundary.

Lemma 1.5.2 *Let D be an operator of Laplace type. Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be the Robin boundary operators defined in Equation* (1.5.c). *Then:*

1. *$(D, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$.*
2. *$\tilde{B}$ defines the adjoint boundary condition for $\tilde{D}$.*

Proof: The map $\mathfrak{b}(\zeta, \lambda)$ of Equation (1.4.h) is now given by

$$\mathfrak{b}(\zeta, \lambda)\phi = -\sqrt{|\zeta|^2 - \lambda}\,\phi\,.$$

Since $|\zeta|^2 - \lambda \neq 0$ for $(0,0) \neq (\zeta, \lambda) \in T^*(\partial M) \times \mathcal{C}$, $\mathfrak{b}$ is an isomorphism so $(D, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$ by Lemma 1.4.8.

To prove Assertion (2), we use Lemma 1.4.17 to compute that

$$\begin{aligned}
\langle D\phi,\rho\rangle_{L^2}-\langle\phi,\tilde D\rho\rangle_{L^2} &= \int_{\partial M}\left\{\langle\phi_{;m}+S\phi,\rho\rangle-\langle\phi,\rho_{;m}+\tilde S\rho\rangle\right\}dy\\
&= \int_{\partial M}\left\{\langle\mathcal{B}\phi,\rho\rangle-\langle\phi,\tilde{\mathcal{B}}\rho\rangle\right\}dy\,.
\end{aligned}$$

The same argument used to establish Assertion (2) of Lemma 1.5.1 completes the proof in the present instance as well. $\square$

More generally, suppose given a decomposition of $\partial M = C_D\,\dot\sqcup\, C_R$ as a disjoint union of closed subsets of ∂M. If we take Dirichlet boundary conditions on C_D and Robin boundary conditions on C_R, the computations performed above show these boundary conditions are elliptic with respect to the cone $\mathcal{C}$. Furthermore as the structures don't interact, the dual boundary conditions are once again Dirichlet boundary conditions on C_D and Robin boundary conditions on C_R.

1.5.3 Mixed boundary conditions

These generalize both Dirichlet and Robin boundary conditions. Assume given an endomorphism χ of $V|_{\partial M}$ so that $\chi^2=\mathrm{Id}_V$. We extend χ to a neighborhood of ∂M in M by requiring that $\chi_{;m}=0$; this preserves the relation $\chi^2=\mathrm{Id}_V$. Let $\tilde\chi$ be the dual endomorphism of V^*. We compute

$$\begin{aligned}
&\langle\phi,\tilde\chi_{;m}\rho\rangle=\langle\phi,[\tilde\nabla_{e_m},\tilde\chi]\rho\rangle=\langle\phi,\tilde\nabla_{e_m}\tilde\chi\rho\rangle-\langle\chi\phi,\tilde\nabla_{e_m}\rho\rangle\\
=\;& e_m\langle\phi,\tilde\chi\rho\rangle-\langle\nabla_{e_m}\phi,\tilde\chi\rho\rangle-e_m\langle\chi\phi,\rho\rangle+\langle\nabla_{e_m}\chi\phi,\rho\rangle\\
=\;& \langle[\nabla_{e_m},\chi]\phi,\rho\rangle=\langle\chi_{;m}\phi,\rho\rangle=0
\end{aligned}$$

and consequently $\tilde\chi_{;m}=0$ as well. Near the boundary, let

$$\begin{aligned}
\Pi_\pm &:= \tfrac12(\mathrm{Id}_V\pm\chi), &\quad V_\pm &:= \Pi_\pm V,\\
\tilde\Pi_\pm &:= \tfrac12(\mathrm{Id}_V\pm\tilde\chi), &\quad V_\pm^* &:= \tilde\Pi_\pm V^*
\end{aligned}$$

be the associated spectral projections and eigenspaces. The pairing $\langle\cdot,\cdot\rangle$ from $V\otimes V^*$ to $\mathbb{R}$ extends to pairings

$$\langle\cdot,\cdot\rangle : V_+\otimes V_+\to\mathbb{R}\quad\text{and}\quad\langle\cdot,\cdot\rangle : V_-\otimes V_-\to\mathbb{R}$$

with the orthogonality relation $V_\pm\perp V_\mp^*$. Since $\chi_{;m}=0$ and $\tilde\chi_{;m}=0$,

$$\Pi_\pm\nabla_{e_m}=\nabla_{e_m}\Pi_\pm,\ \text{and}\ \tilde\Pi_\pm\tilde\nabla_{e_m}=\tilde\nabla_{e_m}\tilde\Pi_\pm\,.$$

Let S be an endomorphism of $V_+|_{\partial M}$. Extend S to $V|_{\partial M}$ as the zero endomorphism of $V_-|_{\partial M}$. Then extend S to a neighborhood of ∂M so that $S_{;m}=0$. Then

$$S\Pi_\pm=\Pi_\pm S,\quad\tilde S\tilde\Pi_\pm=\tilde\Pi_\pm\tilde S,\quad\text{and}\quad\tilde S_{;m}=0\,.$$

We define the mixed boundary operators

$$\begin{aligned} \mathcal{B}\phi &:= \Pi_+(\nabla_{e_m} + S)\phi|_{\partial M} \oplus \Pi_-\phi|_{\partial M}, \\ \tilde{\mathcal{B}}\rho &:= \tilde{\Pi}_+(\tilde{\nabla}_{e_m} + \tilde{S})\rho|_{\partial M} \oplus \tilde{\Pi}_-\rho|_{\partial M}\,. \end{aligned} \tag{1.5.d}$$

Lemma 1.5.3 *Let D be an operator of Laplace type. Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be the mixed boundary operators defined in Equation* (1.5.d). *Then*

1. $(D, \mathcal{B})$ *is elliptic with respect to the cone $\mathcal{C}$.*
2. $\tilde{B}$ *defines the adjoint boundary condition for $\tilde{D}$.*

Proof: On the leading symbol level, the boundary conditions decouple into pure Dirichlet on V_- and pure Neumann on V_+. Thus Assertion (1) follows from Lemma 1.5.1 and Lemma 1.5.2.

To prove Assertion (2), we use Lemma 1.4.17 to show that

$$\begin{aligned} &\int_M \left\{\langle D\phi, \rho\rangle - \langle \phi, \tilde{D}\rho\rangle\right\}dx = \int_{\partial M}\left\{\langle \phi_{;m}, \rho\rangle - \langle \phi, \rho_{;m}\rangle\right\}dy \\ = \;& \int_{\partial M}\left\{\langle \phi_{+;m}, \rho_+\rangle + \langle \phi_{-;m}, \rho_-\rangle - \langle \phi_+, \rho_{+;m}\rangle - \langle \phi_-, \rho_{-;m}\rangle\right\}dy \\ = \;& \int_{\partial M}\left\{\langle \mathcal{B}\phi, \tilde{\Pi}_+\rho - \tilde{\Pi}_-\rho_{;m}\rangle - \langle \Pi_+\phi - \Pi_-\phi_{;m}, \tilde{\mathcal{B}}\rho\rangle\right\}dy\,. \end{aligned}$$

Exactly the same argument as that given to study the adjoint boundary conditions for Dirichlet and Robin boundary conditions now show $\tilde{\mathcal{B}}$ defines the adjoint boundary condition in the mixed setting. □

1.5.4 Absolute boundary conditions

Absolute boundary conditions are a special case of mixed boundary conditions. They arise in index theory [189] and also in the context of quantum gravity, see Luckock and Moss [270]. We take normalized coordinates $x = (y, x_m)$ on a collared neighborhood of the boundary as was discussed in Section 1.1.2. The metric then has the form

$$ds^2 = g_{\alpha\beta}dy^\alpha \circ dy^\beta + dx^m \circ dx^m\,. \tag{1.5.e}$$

If $I = \{1 \le \alpha_1 < \alpha_2 < ... < \alpha_p \le m-1\}$ is a multi-index, one defines

$$dy^I := dy^{\alpha_1} \wedge ... \wedge dy^{\alpha_p}\,.$$

Let ψ be a smooth differential form. Near ∂M, expand

$$\psi = \sum_I \psi_I^+ dy^I + \sum_J \psi_J^- dx^m \wedge dy^J\,.$$

The *absolute boundary operator* $\mathcal{B}_a$ is defined by setting

$$\mathcal{B}_a\psi := \left\{\sum_I \partial_m\psi_I^+|_{\partial M} \cdot dy^I\right\} \oplus \left\{\sum_J \psi_J^-|_{\partial M} \cdot dy^J\right\}. \tag{1.5.f}$$

It is clear that

$$\mathcal{B}_a\psi = 0 \qquad \Rightarrow \qquad \mathcal{B}_a d\psi = 0\,. \tag{1.5.g}$$

We now show that $\mathcal{B}_a$ is invariantly defined. As in Section 1.2, let $\mathfrak{e}$ be exterior multiplication and, dually, let $\mathfrak{i}$ be interior multiplication. Let L be the second fundamental form.

Lemma 1.5.4 *Let $\Delta = d\delta + \delta d$ and let $\mathcal{B}$ the absolute boundary operator defined in Equation* (1.5.f).

1. $\mathcal{B} = \mathcal{B}_{\chi,S}$ *where:*

  (a) $\chi = -1$ *on* $\Lambda(\partial M)^\perp = \operatorname{Span}\{dx^m \wedge dy^I\}$.
  (b) $\chi = +1$ *on* $\Lambda(\partial M) = \operatorname{Span}\{dy^I\}$.
  (c) $S = -\Pi_+\mathfrak{e}(e_a)\mathfrak{i}(e_b)L_{ab}\Pi_+$ *on* $\Lambda(\partial M)$.

2. $\chi_{:a} = 2L_{ab}(\mathfrak{e}_b\mathfrak{i}_m + \mathfrak{e}_m\mathfrak{i}_b)$.

3. $(\Delta, \mathcal{B})$ *is elliptic with respect to the cone* $\mathcal{C}$.

4. $(\Delta, \mathcal{B})$ *is self-adjoint.*

Proof: The form of the metric tensor given in Equation (1.5.e) shows that $\Gamma_m{}^\mu{}_\nu = 0$ if either $\mu = m$ or $\nu = m$. We use Equation (1.1.a) to compute that

$$\begin{aligned}
\nabla_{e_m}(\psi_I^+ dy^I) &= \{\partial_m\psi_I^+ + \psi_I^+\Gamma_m{}^\alpha{}_\beta\mathfrak{e}(dy^\beta)\mathfrak{i}(\partial_\alpha^y)\}dy^I\\
&= \{\partial_m\psi_I^+ - \psi_I^+\Gamma_{m\beta}{}^\alpha\mathfrak{e}(dy^\beta)\mathfrak{i}(\partial_\alpha^y)\}dy^I\\
&= \{\partial_m + L_\beta^\alpha\mathfrak{e}(dy^\beta)\mathfrak{i}(\partial_\alpha^y)\}\psi_I^+ dy^I\,.
\end{aligned}$$

We establish Assertion (1) by using this identity to express

$$\mathcal{B} = \Pi_+\{\nabla_{e_m} - L_\beta^\alpha\mathfrak{e}(dy^\beta)\mathfrak{i}(\partial_\alpha^y)\} \oplus \Pi_-\,.$$

Let $\omega_+ := e^{a_1} \wedge \ldots \wedge e^{a_\ell}$ and $\omega_- := e^m \wedge \omega_+$. As $\chi\omega_\pm = \pm\omega_\pm$, we may establish Assertion (2) by computing

$$\begin{aligned}
(\nabla_{e_a}\chi - \chi\nabla_{e_a})\omega_+ &= (\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b + \Gamma_{abm}\mathfrak{e}_m\mathfrak{i}_b - \Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b + \Gamma_{abm}\mathfrak{e}_m\mathfrak{i}_b)\omega_+\\
&= 2L_{ab}\mathfrak{e}_m\mathfrak{i}_b\omega_+,\\
(\nabla_{e_a}\chi - \chi\nabla_{e_a})\omega_- &= (-\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b - \Gamma_{amb}\mathfrak{e}_b\mathfrak{i}_m + \Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b - \Gamma_{amb}\mathfrak{e}_b\mathfrak{i}_m)\omega_-\\
&= 2L_{ab}\mathfrak{e}_b\mathfrak{i}_m\omega_-\,.
\end{aligned}$$

Assertion (3) now follows from Assertion (1) and from Lemma 1.5.3. By Assertion (1), S is self-adjoint. Assertion (4) follows from Lemma 1.5.3. $\square$

1.5.5 Relative boundary conditions

The Hodge $\star$ operator of Section 1.2.5 is only locally defined if M is not orientable but this minor technical difficulty plays no role in our analysis. *Relative boundary conditions* are defined by dualizing with respect to $\star$ to define

$$\mathcal{B}_r\psi := \mathcal{B}_a(\star\psi)\,. \tag{1.5.h}$$

Consequently

$$\star_p \Delta^p_{\mathcal{B}_a} \star_{m-p} = (-1)^{p(m-p)} \Delta^{m-p}_{\mathcal{B}_r} . \tag{1.5.i}$$

The following reformulation of absolute and relative boundary conditions is a useful one. Near the boundary, we may decompose any smooth form ω into a sum $\omega = \omega_1 + dx^m \wedge \omega_2$ where the ω_i are tangential differential forms and where x_m is the geodesic distance to the boundary. Let i be the inclusion of ∂M into M. We define

$$\mathfrak{C}_a \omega = i^* \mathrm{i}(e_m)\omega = \omega_2|_{\partial M} \quad \text{and} \quad \mathfrak{C}_r \omega = i^* \omega = \omega_1|_{\partial M} . \tag{1.5.j}$$

Let ϕ be an auxiliary smooth function. Let $d_\phi := e^{-\phi} d e^{\phi}$ and $\delta_\phi := e^{\phi}\delta e^{-\phi}$. We set $\phi = 1$ to recover d and δ.

Lemma 1.5.5 *Assume $\phi \in C^\infty(M)$ satisfies Neumann boundary conditions.*

1. *The operator $\mathfrak{C}_a \oplus \mathfrak{C}_a d_\phi$ defines absolute boundary conditions.*
2. *The operator $\mathfrak{C}_r \oplus \mathfrak{C}_r \delta_\phi$ defines relative boundary conditions.*
3. *If $\mathfrak{C}_r \psi = 0$, then $\mathfrak{C}_r d_\phi \psi = 0$.*
4. *If $\mathfrak{C}_a \psi = 0$, then $\mathfrak{C}_a \delta_\phi \psi = 0$.*

Proof: Let d_T be the exterior derivative and let $\star_T$ be the Hodge operator on ∂M. Decompose $\psi = \psi^+_I dy^I + \psi^-_J dx^m \wedge dy^J$. Since $\partial^x_m \phi|_{\partial M} = 0$,

$$\begin{aligned} & \mathfrak{C}_a \psi \oplus \mathfrak{C}_a d_\phi \psi \\ = \quad & \psi^-_J|_{\partial M} dy^J \oplus \{(\partial^x_m \psi^+_I|_{\partial M}) dy^I - e^{-\phi} d_T (e^{\phi} \psi^-_J dy^J|_{\partial M})\} . \end{aligned}$$

Assertion (1) now follows from Equation (1.5.f).

We ignore the precise signs as they play no role in defining the boundary conditions. We have

$$\mathfrak{C}_r = \pm \star_T \mathfrak{C}_a \star \quad \text{and} \quad \delta_\phi = \pm \star d_\phi \star .$$

Assertion (2) now follows dually from Assertion (1) and from Lemma 1.2.7.

We may express $\mathfrak{C}_r d_\phi = \{d_T + \mathfrak{e}(d_T \phi)\}\mathfrak{C}_r$. Since the operator $d_T + \mathfrak{e}(d_T\phi)$ is tangential, $\mathfrak{C}_r \psi = 0$ implies $\mathfrak{C}_r d_\phi \psi = 0$. This establishes Assertion (3); Assertion (4) then follows by duality. $\square$

1.5.6 The de Rham complex and the Hodge decomposition theorem

We now establish a basic technical result. Let $\Delta_\phi := d_\phi \delta_\phi + \delta_\phi d_\phi$ be the Witten Laplacian.

Lemma 1.5.6 *Let $\mathcal{B}$ denote either absolute or relative boundary conditions. Assume ϕ satisfies Neumann boundary conditions. Let $E_\lambda(\Delta^p_{\phi,\mathcal{B}})$ be the associated eigenspaces on p forms. If $\lambda \neq 0$, then we have long exact sequences*

$$0 \to E_\lambda(\Delta^0_{\phi,\mathcal{B}}) \xrightarrow{d_\phi} E_\lambda(\Delta^1_{\phi,\mathcal{B}}) \xrightarrow{d_\phi} \ldots \xrightarrow{d_\phi} E_\lambda(\Delta^{m-1}_{\phi,\mathcal{B}}) \xrightarrow{d_\phi} E_\lambda(\Delta^m_{\phi,\mathcal{B}}) \to 0$$
$$0 \leftarrow E_\lambda(\Delta^0_{\phi,\mathcal{B}}) \xleftarrow{\delta_\phi} E_\lambda(\Delta^1_{\phi,\mathcal{B}}) \xleftarrow{\delta_\phi} \ldots \xleftarrow{\delta_\phi} E_\lambda(\Delta^{m-1}_{\phi,\mathcal{B}}) \xleftarrow{\delta_\phi} E_\lambda(\Delta^m_{\phi,\mathcal{B}}) \leftarrow 0 .$$

Proof: We shall suppose that $\mathcal{B}$ defines absolute boundary conditions; the corresponding assertion for relative boundary conditions will then follow by duality. Let $\lambda \neq 0$ and let $\psi \in E_\lambda(\Delta_{\phi,\mathcal{B}})$; we suppress the index p to simplify the notation. Since $d_\phi\Delta_\phi = \Delta_\phi d_\phi$ and $\delta_\phi\Delta_\phi = \Delta_\phi\delta_\phi$,

$$\Delta_\phi d_\phi\psi = \lambda d_\phi\psi \quad \text{and} \quad \Delta_\phi\delta_\phi\psi = \lambda\delta_\phi\psi\,.$$

Furthermore, since $\delta_\phi d_\phi + d_\phi\delta_\phi = \lambda$, the sequences in question will be exact. Thus to complete the proof, we need only check the boundary conditions are preserved.

We apply Lemma 1.5.5. Since $\mathcal{B}_a\psi = 0$, $\mathfrak{C}_a\psi = 0$ and $\mathfrak{C}_a d_\phi\psi = 0$. Since $d_\phi d_\phi\psi = 0$, we also have $\mathfrak{C}_a d_\phi(d_\phi\psi) = 0$. Thus $\mathcal{B}_a d_\phi\psi = 0$ as desired. As $\mathfrak{C}_a\psi = 0$, $\mathfrak{C}_a\delta_\phi\psi = 0$. We may express

$$\mathfrak{C}_a d_\phi\delta_\phi\psi = \mathfrak{C}_a(\lambda\psi) - \mathfrak{C}_a\delta_\phi d_\phi\psi = \mathfrak{C}_a\delta_\phi d_\phi\psi\,.$$

Since $\mathfrak{C}_a d_\phi\psi = 0$, $\mathfrak{C}_a\delta_\phi d_\phi\psi = 0$. This shows $\mathfrak{C}_a d_\phi\delta_\phi\psi = 0$ so $\mathcal{B}_a\delta_\phi\psi = 0$. $\square$

Let M be a closed Riemannian manifold. The *Hodge decomposition theorem* for a closed Riemannian manifold yields a decomposition of the form

$$C^\infty(\Lambda(M)) = \text{range}\,(d) + \text{range}\,(\delta) + \ker(\Delta)\,.$$

Let $\{\rho_i^+, \lambda_i^+\}$ be a spectral resolution of the Laplacian on range$\,(d)$, $\{\rho_i^-, \lambda_i^-\}$ be a spectral resolution of the Laplacian on range$\,(\delta)$, and $\{\rho_i^0\}$ be an orthonormal basis for $\ker(\Delta)$. This yields a complete orthonormal basis for $L^2(\Lambda(M))$ so that

$$\begin{array}{ll} d\delta\rho_i^+ = \lambda_i^+\rho_i^+, & d\rho_i^+ = 0, \\ \delta\rho_j^- = 0, & \delta d\rho_j^- = \lambda_j^-\rho_j^-, \\ \delta\rho_k^0 = 0, & d\rho_k^0 = 0\,. \end{array}$$

This generalizes to the setting of smooth manifolds with boundary as:

Theorem 1.5.7 *Let $\mathcal{B}$ denote either absolute or relative boundary conditions. There is a complete orthonormal basis $\{\rho_i^+\,\rho_j^-, \rho_k^0\}$ for $L^2(\Lambda(M))$ so that*

$$\begin{array}{lll} d\delta\rho_i^+ = \lambda_i^+\rho_i^+, & d\rho_i^+ = 0, & \mathcal{B}\rho_i^+ = 0, \\ \delta\rho_j^- = 0, & \delta d\rho_j^- = \lambda_j^-\rho_j^-, & \mathcal{B}\rho_j^- = 0, \\ \delta\rho_k^0 = 0, & d\rho_k^0 = 0, & \mathcal{B}\rho_k^0 = 0\,. \end{array}$$

As for closed manifolds, the zero mode eigenspace which is spanned by the differential forms ρ_k^0 has a cohomological interpretation. Let $H^p(M)$ and $H^p(M, \partial M)$ denote the absolute and relative cohomology groups, respectively, with complex coefficients. Theorem 1.2.4 generalizes to this setting to yield the *Hodge-de Rham isomorphism*

Theorem 1.5.8 (Hodge-de Rham) *Suppose that M is a compact Riemannian manifold with smooth boundary. Then there are natural isomorphisms $E_0(\Delta^p_{\mathcal{B}_a}) \approx H^p(M)$ and $E_0(\Delta^p_{\mathcal{B}_r}) \approx H^p(M, \partial M)$.*

Remark 1.5.9 If M is orientable, then the Hodge $\star$ operator realizes the *Poincaré duality isomorphism*

$$\star : H^p(M) = E_0(\Delta^p_{\mathcal{B}_a}) \xrightarrow{\approx} E_0(\Delta^{m-p}_{\mathcal{B}_r}) = H^{m-p}(M, \partial M) . \tag{1.5.k}$$

Lemma 1.3.10 generalizes to this setting to yield the following result.

Lemma 1.5.10 *Suppose that M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}_a$ (resp. $\mathcal{B}_r$) denote absolute (resp. relative) boundary conditions. Let ϕ satisfy Neumann boundary conditions. Then*

$$\sum_p (-1)^p a_n(1, \Delta^p_\phi, \mathcal{B}) = \begin{cases} \chi(M) & \text{if } n = m,\ \mathcal{B} = \mathcal{B}_a, \\ \chi(M, \partial M) & \text{if } n = m,\ \mathcal{B} = \mathcal{B}_r, \\ 0 & \text{if } n \neq m,\ \mathcal{B} = \mathcal{B}_a, \mathcal{B}_r . \end{cases}$$

Proof: By Lemma 1.5.6, the same Bott cancellation argument as that used above to establish Lemma 1.3.9 now shows that

$$\sum_p a_n(1, \Delta^p_\phi, \mathcal{B}) = \begin{cases} 0 & \text{if} \quad n \neq m, \\ \sum_p (-1)^p \dim\ker(\Delta^p_{\phi,\mathcal{B}_a}) & \text{if} \quad n = m . \end{cases}$$

Thus the supertrace heat asymptotics vanish if $n \neq m$.

Suppose that $n = m$. Since $\sum_p (-1)^p a_m(1, \Delta^p_\phi, \mathcal{B})$ is given by a local formulae, it is a continuous integer valued function of ϕ and hence constant. Thus by considering the smooth 1 parameter family $\phi_\varepsilon := \varepsilon\phi$, we may without loss of generality assume $\phi = 0$ in evaluating the supertrace; the value of the supertrace for $n = m$ can now be evaluated using Theorem 1.5.8. $\square$

Remark 1.5.11 We will show subsequently in Section 3.8 that the supertrace

$$\sum_{p=0}^{m} (-1)^p a_n(1, \Delta^p_\phi, \mathcal{B})$$

is independent of ϕ for $n \leq m$ and thus the assumption that ϕ satisfies Neumann boundary conditions is unnecessary in studying the index contribution. On the other hand, taking $m = 1$ and $M = [0, \pi]$, we shall show that

$$a_2(1, \Delta^0_\phi, \mathcal{B}) - a_2(1, \Delta^1_\phi) = \tfrac{1}{\sqrt{\pi}} \int_M \phi_{;11} dx .$$

This can be non-zero if ϕ does not satisfy Neumann boundary conditions. Thus the vanishing can fail for $n > m$ for general ϕ.

Example 1.5.12 Let $M = [0, \pi]$. Then

$$\Delta^0 f = -\partial_x^2 f \quad \text{and} \quad \Delta^1(f \cdot dx) = -\partial_x^2 f \cdot dx .$$

Absolute boundary conditions correspond to pure Neumann boundary conditions for Δ^0 and Dirichlet boundary conditions for Δ^1. The corresponding spectral resolutions are then given by

$$\begin{aligned} \left\{ \tfrac{1}{\sqrt{\pi}}, 0 \right\} \cup \left\{ \tfrac{\sqrt{2}}{\sqrt{\pi}} \cos nx, n^2 \right\}_{n=1}^{\infty} &\quad \text{for } \Delta^0_{\mathcal{B}_a}, \\ \left\{ \tfrac{\sqrt{2}}{\sqrt{\pi}} \sin nx \cdot dx, n^2 \right\}_{n=1}^{\infty} &\quad \text{for } \Delta^1_{\mathcal{B}_a} . \end{aligned} \tag{1.5.l}$$

Poincaré duality interchanges the roles of absolute and relative boundary conditions. Thus relative boundary conditions are Dirichlet boundary conditions for Δ^0 and Neumann boundary conditions for Δ^1. The corresponding spectral resolutions take the form

$$\begin{aligned} \left\{\tfrac{\sqrt{2}}{\sqrt{\pi}}\sin nx, n^2\right\}_{n=1}^{\infty} &\quad \text{for } \Delta^0_{\mathcal{B}_r}, \\ \left\{\tfrac{1}{\sqrt{\pi}}dx, 0\right\} \cup \left\{\tfrac{\sqrt{2}}{\sqrt{\pi}}\cos nx\cdot dx, n^2\right\}_{n=1}^{\infty} &\quad \text{for } \Delta^1_{\mathcal{B}_r}. \end{aligned} \tag{1.5.m}$$

Thus, in particular,

$$\begin{aligned} &H^0([0,\pi]) = \ker \Delta^0_{\mathcal{B}_a} = \mathbb{C}, && H^1([0,\pi]) = \ker\Delta^1_{\mathcal{B}_a} = 0, \\ &H^0([0,\pi],\partial[0,\pi]) = \ker\Delta^0_{\mathcal{B}_r} = 0, && H^1([0,\pi],\partial[0,\pi]) = \ker\Delta^1_{\mathcal{B}_r} = \mathbb{C}. \end{aligned}$$

Example 1.5.13 We suppose that the metric is a product near the boundary of a manifold M, i.e. that $g_{\alpha\beta} = g_{\alpha\beta}(y)$ is independent of the radial parameter x_m in Equation (1.5.e). We may then take two copies $M_\pm$ of the manifold M and double M setting $N := M_+ \cup_{\partial M} M_-$. The metrics $g_\pm$ then glue smoothly together to define a Riemannian metric on N. Let $\mathcal{F}x_\pm = x_\mp$ denote the flip which interchanges the two factors. As $\mathcal{F}$ is an isometry, we may decompose

$$\begin{aligned} &E_\lambda(\Delta^p_N) = E_\lambda(\Delta^p_N)^+ \oplus E_\lambda(\Delta^p_N)^- \text{ where} \\ &E_\lambda(\Delta^p_N)^\pm := \{\phi \in C^\infty(\Lambda^p(N)) : \Delta^p_N\phi = \lambda\phi \text{ and } \mathcal{F}\phi = \pm\phi\}. \end{aligned}$$

Let $i : M \to M_+ \subset N$ be the natural inclusion. We then have

1. i^* is an isomorphism from $E_\lambda(\Delta^p_N)^+$ to $E_\lambda(\Delta^p_{M,\mathcal{B}_a})$.
2. i^* is an isomorphism from $E_\lambda(\Delta^p_N)^-$ to $E_\lambda(\Delta^p_{M,\mathcal{B}_r})$.

If we decompose

$$\begin{aligned} &H^p(N) = H^p(N)^+ \oplus H^p(N)^- \quad \text{where} \\ &H^p(N)^\pm := \{\Xi \in H^p(N) : \mathcal{F}^*\Xi = \pm\Xi\} \quad \text{then} \\ &H^p(M) = H^p(N)^+ \quad \text{and} \quad H^p(M,\partial M) = H^p(N)^-. \end{aligned}$$

Example 1.5.14 If $M = [0,\pi]$ is the interval of Example 1.5.12, then the double $N = S^1$ is the circle. The spectral resolutions of the Laplacian for N are given by

$$\begin{aligned} \left\{\tfrac{1}{\sqrt{2\pi}}e^{\sqrt{-1}n\theta}, n^2\right\}_{n=-\infty}^{\infty} &\quad \text{for } \Delta^0_{S^1}, \\ \left\{\tfrac{1}{\sqrt{2\pi}}e^{\sqrt{-1}n\theta}d\theta, n^2\right\}_{n=-\infty}^{\infty} &\quad \text{for } \Delta^1_{S^1}. \end{aligned}$$

We use Example 1.5.13 to recover the spectral resolutions given in Example 1.5.12 for the Laplacian with absolute boundary and relative boundary conditions on M. We first note that

$$\begin{aligned} &e^{\sqrt{-1}n\theta} + e^{-\sqrt{-1}n\theta} = 2\cos(n\theta), \quad \text{and} \\ &(e^{\sqrt{-1}n\theta} - e^{-\sqrt{-1}n\theta})d\theta = 2\sqrt{-1}\sin(n\theta)d\theta \end{aligned}$$

are even in θ. Thus they induce the spectral resolution of the Laplacian with absolute boundary conditions on M given in Equation (1.5.l). Similarly as

$$e^{\sqrt{-1}n\theta} - e^{-\sqrt{-1}n\theta} = 2\sqrt{-1}\sin(n\theta), \quad \text{and}$$
$$(e^{\sqrt{-1}n\theta} + e^{-\sqrt{-1}n\theta})d\theta = 2\cos(n\theta)d\theta$$

are odd in θ, they induce the spectral resolution of the Laplacian with relative boundary conditions on M given in Equation (1.5.m).

1.6 Boundary conditions II

We continue our discussion of boundary conditions with some slightly more exotic examples.

1.6.1 Transmission boundary conditions

These boundary conditions appear in the so-called brane-world scenario; see, for example, [77, 286]. The perhaps somewhat perverse sign convention for U that we shall adopt here is motivated by the notation established there.

Assume given two Riemannian manifolds $(M_\pm, g_\pm)$ with common boundary

$$\Sigma := \partial M_+ = \partial M_- \quad \text{with} \quad g_+|_\Sigma = g_-|_\Sigma;$$

no matching condition is assumed on the normal derivatives of the metrics. Let dy be the Riemannian measure on Σ. We let $\nu_\pm$ be the inward unit normals of Σ in $M_\pm$. We note that

$$\nu_+ + \nu_- = 0\,.$$

Let $V_\pm$ be vector bundles over $M_\pm$; we also suppose that

$$V_\Sigma := V_+|_\Sigma = V_-|_\Sigma\,.$$

Let $D_\pm$ be operators of Laplace type on $V_\pm$; no compatibility conditions are placed on these operators. We use Lemma 1.2.1 to express

$$D_\pm = D(\nabla_\pm, E_\pm)\,;$$

in the interest of notational simplicity, we shall let ∇ denote $\nabla_\pm$ on $C^\infty(V_\pm)$. Let U be an auxiliary endomorphism of V_Σ. The *transmission boundary operators* $\mathcal{B}_U$ and $\mathcal{B}_{\tilde{U}}$ are defined for

$$\phi = (\phi_+, \phi_-) \quad \text{and} \quad \rho = (\rho_+, \rho_-)$$

by setting

$$\begin{aligned} \mathcal{B}_U\phi &:= \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \;\oplus\; \{\nabla_{\nu_+}\phi_+|_\Sigma + \nabla_{\nu_-}\phi_-|_\Sigma - U\phi_+|_\Sigma\}, \\ \mathcal{B}_{\tilde{U}}\rho &:= \{\rho_+|_\Sigma - \rho_-|_\Sigma\} \;\oplus\; \{\tilde{\nabla}_{\nu_+}\rho_+|_\Sigma + \tilde{\nabla}_{\nu_-}\rho|_\Sigma - \tilde{U}\rho_+|_\Sigma\}\,. \end{aligned} \tag{1.6.a}$$

Lemma 1.6.1 *Let $D = (D_+, D_-)$ be an operator of Laplace type on the manifold $M = M_+ \cup_\Sigma M_-$. Let $\mathcal{B}_U$ and $\mathcal{B}_{\tilde U}$ be the boundary operators defined in Equation* (1.6.a). *Then*

1. *$(D, \mathcal{B}_U)$ is elliptic with respect to the cone $\mathcal{C}$.*
2. *$\mathcal{B}_{\tilde U}$ defines the adjoint boundary condition for $\tilde D$ on $C^\infty(V^*)$.*

Proof: We must substitute

$$\phi_{+;\nu_+} \to \sqrt{|\zeta|^2 - \lambda} \cdot \phi_+ \text{ and } \phi_{-;\nu_-} \to \sqrt{|\zeta|^2 - \lambda} \cdot \phi_-$$

since $\nu_\pm$ are the relevant inward unit normals. Thus the operator $\mathfrak{b}$ of Equation (1.4.h) becomes

$$\mathfrak{b}(\zeta, \lambda) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} \phi_+ - \phi_- \\ -\sqrt{|\zeta|^2 - \lambda} \cdot (\phi_+ + \phi_-) \end{pmatrix}.$$

As $|\zeta|^2 - \lambda \neq 0$, $\mathfrak{b}$ is an isomorphism. Consequently, Assertion (1) follows from Lemma 1.4.8.

The Green's formula given in Lemma 1.4.16 for this example yields

$$\begin{aligned} &\int_M \left\{ \langle D\phi, \rho \rangle - \langle \phi, \tilde D \rho \rangle \right\} dx \qquad (1.6.b) \\ = &\int_\Sigma \left\{ \langle \phi_{+;\nu_+}, \rho_+ \rangle + \langle \phi_{-;\nu_-}, \rho_- \rangle - \langle \phi_+, \rho_{+;\nu_+} \rangle - \langle \phi_-, \rho_{-;\nu_-} \rangle \right\} dy \\ = &\int_\Sigma \Big\{ \langle \phi_{+;\nu_+} + \phi_{-;\nu_-} - U\phi_+, \rho_+ \rangle + \langle \phi_{-;\nu_-}, \rho_- - \rho_+ \rangle \\ &\quad -\langle \phi_+, \rho_{+;\nu_+} + \rho_{-;\nu_-} - \tilde U \rho_+ \rangle - \langle \phi_- - \phi_+, \rho_{-;\nu_-} \rangle \Big\} dy\,. \end{aligned}$$

Consequently if $\mathcal{B}_U \phi = 0$ and if $\mathcal{B}_{\tilde U} \rho = 0$, then $\langle D\phi, \rho \rangle_{L^2} - \langle \phi, \tilde D \rho \rangle_{L^2} = 0$. This establishes the first condition of Definition 1.4.12. Conversely, if Display (1.6.b) vanishes for all ϕ with $\mathcal{B}_U \phi = 0$, we then have only the term

$$\int_\Sigma \left\{ \langle \phi_+, \rho_{+;\nu_+} + \rho_{-;\nu_-} - \tilde U \rho_+ \rangle + \langle \phi_{-;\nu_-}, \rho_- - \rho_+ \rangle \right\} dy$$

for all such ρ. We may now conclude $\mathcal{B}_{\tilde U} \rho = 0$ since, by Lemma 1.4.1, we can specify ϕ_+ and $\phi_{-;\nu_-}$ arbitrarily on Σ subject to the condition that $\mathcal{B}_U \phi = 0$. This establishes the second condition of Definition 1.4.12 and shows that $\mathcal{B}_{\tilde U}$ defines the adjoint boundary condition. □

Remark 1.6.2 We use the geodesic flow to identify a neighborhood of Σ in M_+ with $\Sigma \times [0, \varepsilon)$ and a neighborhood of Σ in M_- with $\Sigma \times (-\varepsilon, 0]$ for some $\varepsilon > 0$ so that the curves $t \to (y, t)$ are unit speed geodesics normal to the boundary $\Sigma := \Sigma \times \{0\}$. We define a canonical smooth structure on $M = M_+ \cup M_-$ by glueing along $\Sigma \times \{0\}$. The metric then takes the form

$$ds^2 = g_{\pm,\alpha\beta}(y, t) dy^\alpha \circ dy^\beta + dt \circ dt\,.$$

We remark that this metric need not be smooth on M.

We can use U to define a canonical C^1 structure on V. Let s_Σ be a local frame for $V|_\Sigma$. We use parallel transport along the geodesic normals to define a local frame s_- for V_- near Σ so $s_{;\nu_-} = 0$. We extend U to V_+ so $U_{;\nu_+} = 0$ and define s_+ on V_+ near Σ so $s_+ = s_-$ on Σ and so $s_{+;\nu_+} = -Us_+$. We then glue s_+ to s_- over Σ to define a C^1 structure for V over M which is characterized by the property that $\mathcal{B}_U\phi = 0$ if and only if $\phi \in C^1(V)$.

1.6.2 Transmission boundary conditions for the de Rham complex

We now construct *transmission boundary conditions for the de Rham complex* which are analogous to the absolute and relative boundary conditions discussed in Section 1.5.4. We postpone until Lemma 3.2.3 a discussion of the related index problem.

Let $M = M_+ \cup_\Sigma M_-$. We use Remark 1.6.2 to define a canonical smooth structure on M and to identify $\Lambda(M_+)|_\Sigma$ with $\Lambda(M_-)|_\Sigma$. Let $\phi = (\phi_+, \phi_-)$ where $\phi_\pm$ are smooth differential forms over $M_\pm$. Let

$$\begin{aligned} \mathcal{B}_0\phi &:= \phi_+|_\Sigma - \phi_-|_\Sigma \in C^\infty(\Lambda(M)|_\Sigma), \\ \mathcal{B}_1\phi &:= \mathcal{B}_0\phi \oplus \mathcal{B}_0(d+\delta)\phi\,. \end{aligned} \tag{1.6.c}$$

If $\{e_i\}$ is a local orthonormal frame for $TM|_\Sigma$, let

$$\mathfrak{e}_j := \mathfrak{e}(e_j), \quad \mathfrak{i}_j := \mathfrak{i}(e_j), \quad \text{and} \quad \gamma_i = \mathfrak{e}_j - \mathfrak{i}_j$$

be defined by left exterior multiplication, left interior multiplication, and left Clifford multiplication. We normalize the frame field so e_m is the inward unit normal of $\Sigma \subset M_+$.

Lemma 1.6.3 *Adopt the notation established above.*

1. *Let* $\omega_a = \nabla^+_{e_a} - \nabla^-_{e_a}$. *Then* $\omega_a = (L^+_{ab} + L^-_{ab})(\mathfrak{e}_m\mathfrak{i}_b + \mathfrak{i}_m\mathfrak{e}_b)$.
2. *Let* $U := \gamma_m\gamma_a\omega_a$. *Then* $\mathcal{B}_1$ *and* $\mathcal{B}_U$ *define the same boundary conditions.*
3. $U = (L^+_{ab} + L^-_{ab})(\mathfrak{e}_m\mathfrak{i}_m\mathfrak{i}_a\mathfrak{e}_b + \mathfrak{i}_m\mathfrak{e}_m\mathfrak{e}_a\mathfrak{i}_b)$.

Proof: Since e_m is the inward unit normal of $\Sigma \subset M_+$ and the outward unit normal of $\Sigma \subset M_-$, we have that

$$\Gamma^+_{abm} = -\Gamma^+_{amb} = L^+_{ab} \quad \text{and} \quad \Gamma^-_{abm} = -\Gamma^-_{amb} = -L^-_{ab}\,. \tag{1.6.d}$$

We use Equation (1.2.e) to derive the relations

$$\mathfrak{e}_i\mathfrak{e}_j + \mathfrak{e}_j\mathfrak{e}_i = 0, \quad \mathfrak{i}_i\mathfrak{e}_j + \mathfrak{e}_j\mathfrak{i}_i = \delta_{ij}, \quad \text{and} \quad \mathfrak{i}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{i}_i = 0. \tag{1.6.e}$$

We prove Assertion (1) by using Equations (1.6.d) and (1.6.e) to compute

$$\begin{aligned} \omega_a : &= \nabla^+_{e_a} - \nabla^-_{e_a} = (\Gamma^+_{ak\ell} - \Gamma^-_{ak\ell})\mathfrak{e}_\ell\mathfrak{i}_k \\ &= (\Gamma^+_{abm} - \Gamma^-_{abm})\mathfrak{e}_m\mathfrak{i}_b + (\Gamma^+_{am\ell} - \Gamma^-_{am\ell})\mathfrak{e}_\ell\mathfrak{i}_m \\ &= (L^+_{ab} + L^-_{ab})(\mathfrak{e}_m\mathfrak{i}_b + \mathfrak{i}_m\mathfrak{e}_b)\,. \end{aligned}$$

Let $\mathcal{B}_0\phi = 0$. We use Lemma 1.2.5 to see

$$\{(d+\delta)\phi_+\}|_\Sigma - \{(d+\delta)\phi_-\}|_\Sigma = \gamma_i\{\phi_{+;e_i}|_\Sigma - \phi_{-;e_i}|_\Sigma\}$$

$$= \quad \gamma_m\{\phi_{+;\nu_+}|_\Sigma + \phi_{-;\nu_-}|_\Sigma - \gamma_m\gamma_a\omega_a\phi|_\Sigma\}\,.$$

Since γ_m is an isomorphism, this vanishes if and only if ϕ satisfies the transmission boundary condition defined by U. Assertion (2) now follows.

We complete the proof by applying Equations (1.6.d) and (1.6.e) once again to see that

$$\begin{aligned} U &= (L^+_{ab}+L^-_{ab})(\mathfrak{e}_m-\mathfrak{i}_m)(\mathfrak{e}_a-\mathfrak{i}_a)(\mathfrak{e}_m\mathfrak{i}_b+\mathfrak{i}_m\mathfrak{e}_b)\\ &= (L^+_{ab}+L^-_{ab})(\mathfrak{e}_m\mathfrak{i}_m\mathfrak{i}_a\mathfrak{e}_b+\mathfrak{i}_m\mathfrak{e}_m\mathfrak{e}_a\mathfrak{i}_b). \qquad \square \end{aligned}$$

1.6.3 Transfer boundary conditions

As we did when considering with transmission boundary conditions, we assume given Riemannian manifolds $(M_\pm, g_\pm)$ with $g_+|_\Sigma = g_-|_\Sigma$. As before, no matching condition is assumed on the normal derivatives of the metrics.

We assume given bundles $V_\pm$ over $M_\pm$. We do not assume, however, an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$; indeed, we permit $\dim(V_+) \neq \dim(V_-)$ for example. Instead, we assume given a collection $\mathcal{S}$ of endomorphisms

$$\begin{array}{ll} S_{++}: V_+|_\Sigma \to V_+|_\Sigma, & S_{+-}: V_-|_\Sigma \to V_+|_\Sigma,\\ S_{-+}: V_+|_\Sigma \to V_-|_\Sigma, & S_{--}: V_-|_\Sigma \to V_-|_\Sigma. \end{array}$$

We may then define the boundary operators $\mathcal{B}_\mathcal{S}$ and $\mathcal{B}_{\tilde{\mathcal{S}}}$ by setting

$$\begin{aligned} \mathcal{B}_\mathcal{S}\phi &:= \left\{\begin{pmatrix} \nabla^+_{\nu_+}+S_{++} & S_{+-}\\ S_{-+} & \nabla^-_{\nu_-}+S_{--}\end{pmatrix}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}\right\}\Bigg|_\Sigma,\\ \mathcal{B}_{\tilde{\mathcal{S}}}\rho &:= \left\{\begin{pmatrix} \tilde\nabla^+_{\nu_+}+\tilde S_{++} & \tilde S_{-+}\\ \tilde S_{+-} & \tilde\nabla^-_{\nu_-}+\tilde S_{--}\end{pmatrix}\begin{pmatrix}\rho_+\\ \rho_-\end{pmatrix}\right\}\Bigg|_\Sigma. \end{aligned} \tag{1.6.f}$$

Lemma 1.6.4 *Let $D = (D_+, D_-)$ be an operator of Laplace type on the manifold $M = M_+ \cup_\Sigma M_-$. Let $\mathcal{B}_\mathcal{S}$ and $\mathcal{B}_{\tilde{\mathcal{S}}}$ be the boundary operators defined in Equation* (1.6.f)*. Then*

1. *$(D, \mathcal{B}_\mathcal{S})$ is elliptic with respect to the cone $\mathcal{C}$.*
2. *$\mathcal{B}_{\tilde{\mathcal{S}}}$ defines the adjoint boundary condition for $\tilde D$.*

Proof: If $S_{+-} = S_{-+} = 0$, then Equation (1.6.f) decouples to define Robin boundary conditions for the operators $D_\pm$ on the manifolds $M_\pm$ separately. Since the ellipticity condition does not depend upon the lower order terms, Assertion (1) follows from Lemma 1.5.2.

We use the the Green's formula given in Lemma 1.4.17 to compute that

$$\begin{aligned} &\langle D\phi,\rho\rangle_{L^2} - \langle\phi,\tilde D\rho\rangle_{L^2}\\ =\ &\int_\Sigma \Big\{\langle\phi_{+;\nu_+},\rho_+\rangle + \langle\phi_{-;\nu_-},\rho_-\rangle - \langle\phi_+,\rho_{+;\nu_+}\rangle - \langle\phi_-,\rho_{-;\nu_-}\rangle\Big\}dy\\ =\ &\int_\Sigma \Big\{\langle\phi_{+;\nu_+}+S_{++}\phi_+ + S_{+-}\phi_-,\rho_+\rangle\\ &\quad +\langle\phi_{-;\nu_-}+S_{-+}\phi_+ + S_{--}\phi_-,\rho_-\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad -\langle\phi_+, \rho_{+;\nu_+} + \tilde{S}_{++}\rho_+ + \tilde{S}_{-+}\rho_-\rangle \\
&\quad \left. -\langle\phi_-, \rho_{-;\nu_-} + \tilde{S}_{+-}\rho_+ + \tilde{S}_{--}\rho_-\rangle \right\} dy \\
= &\int_\Sigma \left\{ \langle\mathcal{B}_\mathcal{S}\phi, \rho\rangle - \langle\phi, \mathcal{B}_{\tilde{\mathcal{S}}}\rho\rangle \right\} dy\,.
\end{aligned}
$$

This vanishes if $\mathcal{B}_\mathcal{S}\phi = 0$ and $\mathcal{B}_{\tilde{\mathcal{S}}}\rho = 0$ so Definition 1.4.12 (1) holds. Conversely, suppose that

$$\langle D\phi, \rho\rangle_{L^2} - \langle\phi, \tilde{D}\rho\rangle_{L^2} = 0$$

for all ϕ with $\mathcal{B}_\mathcal{S}\phi = 0$. By Lemma 1.4.1, we can choose ϕ so $\mathcal{B}_\mathcal{S}\phi = 0$ and so $\phi|_\Sigma$ is arbitrary. It now follows that $\mathcal{B}_{\tilde{\mathcal{S}}}\rho = 0$ so Definition 1.4.12 (2) is satisfied. $\square$

1.6.4 Heat conduction problems

The boundary conditions defined by Equations (1.6.a) and (1.6.f) can be regarded as living on the singular manifold $M := M_+ \cup_\Sigma M_-$ and are relevant to heat transfer problems between the two media M_+ and M_-. Whether to use transfer or transmission boundary conditions depends on nature of the separation between M_+ and M_-.

We follow the discussion in Carslaw and Jaeger [116]. As the heat flux is continuous over the interface Σ, the heat flowing out of (or into) M_+ is equal to the heat flowing into (or out of) M_-. This means that

$$\phi_{+;\nu_+}|_\Sigma + \phi_{-;\nu_-}|_\Sigma = 0\,. \tag{1.6.g}$$

If the contact between M_+ and M_- is very close, then one has

$$\phi_+|_\Sigma = \phi_-|_\Sigma\,.$$

This yields the boundary condition defined by the boundary operator of Equation (1.6.a) with impedance matching term $U = 0$. Otherwise, if M_+ and M_- are only pressed together lightly, then the best linear approximation yields that the flux of heat between M_+ and M_- is proportional to their temperature difference. This yields the additional relation

$$\phi_{+;\nu_+}|_\Sigma = H(\phi_+ - \phi_-)|_\Sigma \tag{1.6.h}$$

where H is the *surface conductivity*. The boundary conditions of Equations (1.6.g) and (1.6.h) are then defined by a boundary operator of the form given in Equation (1.6.f) where $S_{++} = S_{--} = -H$ and $S_{+-} = S_{-+} = H$.

1.6.5 Bag boundary conditions

We follow the discussion in [33]. Let A be an operator of Dirac type on a bundle V over an oriented Riemannian manifold M of even dimension $m = 2\bar{m}$. Let $\{e_1, ..., e_m\}$ be an oriented local orthonormal basis for the tangent bundle.

We expand $A = \gamma_i \nabla_{e_i} + \psi_A$. The *chiral operator* discussed in Section 1.1.4 is defined by

$$\gamma_{m+1} := (\sqrt{-1})^{\frac{m}{2}} \gamma_1 \dots \gamma_m \,.$$

We note that $\gamma_{m+1}\gamma_{m+1} = \mathrm{Id}_V$ and that γ_{m+1} anti-commutes with γ_i for $i \leq m$. Let θ be a real parameter. We define

$$\begin{aligned}
&\chi_\theta := \gamma_{m+1} e^{\theta\gamma_{m+1}} \gamma_m = \gamma_{m+1}(\cosh\theta + \sinh\theta\gamma_{m+1})\gamma_m, \\
&\Pi_\theta^\pm := \tfrac{1}{2}(\mathrm{Id}_V \pm \chi_\theta), \quad \text{and} \qquad\qquad (1.6.\mathrm{i}) \\
&\mathcal{B}_\theta\phi := \Pi_\theta^+ \phi|_{\partial M} \oplus \Pi_\theta^+ A\phi|_{\partial M} \,.
\end{aligned}$$

Since $\gamma_{m+1}\gamma_m = -\gamma_m\gamma_{m+1}$, we have that

$$\begin{aligned}
\chi_\theta^2 &= \gamma_{m+1} e^{\theta\gamma_{m+1}} \gamma_m \gamma_{m+1} e^{\theta\gamma_{m+1}} \gamma_m \\
&= -\gamma_{m+1}\gamma_{m+1} e^{\theta\gamma_{m+1}} e^{-\theta\gamma_{m+1}} \gamma_m\gamma_m \\
&= \mathrm{Id}_V \,.
\end{aligned}$$

Thus $\Pi_\theta^\pm$ is projection on the ± 1 eigenspace of χ_θ. Since γ_m anti-commutes with χ_0, $\gamma_m \Pi_0^\pm (\gamma_m)^{-1} = \Pi_0^\mp$ so

$$\dim \mathrm{range}\,(\Pi_0^+) = \dim \mathrm{range}\,(\Pi_0^-) = \tfrac{1}{2}\dim V \,.$$

Since $\Pi_\theta^\pm$ are smooth 1 parameter families of projections, $\dim\mathrm{range}\,(\Pi_\theta^\pm)$ is independent of θ and consequently

$$\dim \mathrm{range}\,(\Pi_\theta^\pm) = \tfrac{1}{2}\dim V \quad \text{for all} \quad \theta \in \mathbb{R}. \qquad (1.6.\mathrm{j})$$

Consequently Π_θ^+ defines an admissible 0^{th} order boundary operator for A; $\mathcal{B}_\theta$ is then the associated boundary operator for A^2.

Lemma 1.6.5 *Let A be an operator of Dirac type. Let Π_θ^+ and $\mathcal{B}_\theta$ be as defined in Equation* (1.6.i)*.*

1. *(A, Π_θ^+) is elliptic with respect to the cone $\mathcal{K}$.*
2. *$(A^2, \mathcal{B}_\theta)$ is elliptic with respect to the cone $\mathcal{C}$.*
3. *$\tilde{\Pi}_{-\theta}^+$ defines the adjoint boundary condition for $\tilde{A}$ on V^*.*

Proof: We apply Lemma 1.4.9 to prove Assertion (1). We recall some notation defined previously in Equations (1.4.i) and (1.4.k). For $(0,0) \neq (\zeta,\lambda)$ in $T^*(\partial M) \times \mathcal{K}$, let

$$\begin{aligned}
&\Xi(\zeta,\lambda) := \sqrt{-1}\gamma_m\gamma_a\zeta_a - \gamma_m\lambda, \quad \text{and} \\
&V_\pm(\zeta,\lambda) := \{v \in V|_{\partial M} : \Xi(\zeta,\lambda)v = \pm\sqrt{|\zeta|^2 - \lambda^2}\cdot v\} \,.
\end{aligned}$$

To show that $\Pi_\theta^+ : V_\pm(\zeta,\lambda) \to \mathrm{range}\,(\Pi_\theta^+)$ is an isomorphism, it suffices, by Equation (1.6.j), to show that

$$\ker \Pi_\theta^+ \cap V_\pm(\zeta,\lambda) = \{0\} \,.$$

Since, by definition, $\ker\Pi_\theta^+ = \mathrm{range}\,\Pi_\theta^-$, we must show

$$\begin{aligned}
&\mathrm{range}\,\Pi_\theta^\pm \cap V_+(\zeta,\lambda) = \{0\}, \quad \text{and} \\
&\mathrm{range}\,\Pi_\theta^\pm \cap V_-(\zeta,\lambda) = \{0\} \,.
\end{aligned}$$

This is equivalent to showing that $\Pi_\theta^\pm$ and $\Xi(\zeta, \lambda)$ do not have a non-trivial joint eigenvector.

By changing coordinates, we may suppose that $\zeta_a dy^a = \tilde{\zeta}_1 d\tilde{y}^1$. Thus without loss of generality, we may suppose $\zeta_2 = ... = \zeta_{m-1} = 0$. We change notation slightly from that established earlier in the proof of Lemma 1.1.5 and introduce the endomorphisms

$$\tau_1 := \sqrt{-1}\gamma_2\gamma_3, \; ... \; \tau_{\bar{m}-1} := \sqrt{-1}\gamma_{m-2}\gamma_{m-1} \, .$$

For $1 \le i, j \le \bar{m} - 1$, one has that

$$\begin{aligned} \tau_i\tau_j &= \tau_j\tau_i, \quad \tau_i^2 = \mathrm{Id}_V, \\ \tau_i\gamma_1 &= \gamma_1\tau_i, \quad \tau_i\gamma_m = \gamma_m\tau_i \, . \end{aligned}$$

Let $\vec{\rho} := (\rho_1, ..., \rho_{\bar{m}-1})$ be a collection of signs $\rho_i = \pm 1$. We decompose

$$V = \oplus_{\vec{\rho}} V_{\vec{\rho}} \quad \text{where} \quad V_{\vec{\rho}} = \{v \in V : \tau_i v = \rho_i v \text{ for } 1 \le i \le \bar{m} - 1\} \, .$$

Since the endomorphisms $\Xi(\zeta, \lambda)$, χ_θ, γ_1, and γ_m preserve each $V_{\vec{\rho}}$, the problem decouples and we may study each simultaneous eigenspace $V_{\vec{\varrho}}$ separately. Let $\tau_{\bar{m}} := \sqrt{-1}\gamma_1\gamma_m$. We further decompose

$$V_{\vec{\rho}} = V_{\vec{\rho}}^+ \oplus V_{\vec{\rho}}^-$$

into the ± 1 eigenspaces of $\tau_{\bar{m}}$. Let $\{u_1, ..., u_\ell\}$ be a basis for $V_{\vec{\rho}}^+$. Then $\{\gamma_m u_1, ..., \gamma_m u_\ell\}$ is a basis for $V_{\vec{\rho}}^-$. Relative to the basis

$$\{u_1, \gamma_m u_1, ..., u_\ell, \gamma_m u_\ell\},$$

we can represent the action of γ_m, $\tau_{\bar{m}}$, γ_1, γ_{m+1}, χ_θ, and Ξ as 2×2 blocks of the form

$$\begin{aligned}
&\tau_{\bar{m}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
&\gamma_1 = \sqrt{-1}\tau_{\bar{m}}\gamma_m = -\sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\gamma_{m+1} = \tau_1 ... \tau_{\bar{m}} = \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{for} \quad \delta = \varrho_1 ... \varrho_{\bar{m}-1}, \\
&\chi_\theta = \begin{pmatrix} 0 & -\delta e^{\delta\theta} \\ -\delta e^{-\delta\theta} & 0 \end{pmatrix} \quad \text{and} \quad \Xi(\zeta, \lambda) = \begin{pmatrix} -\zeta_1 & \lambda \\ -\lambda & \zeta_1 \end{pmatrix} .
\end{aligned}$$

The eigenvectors of χ_θ and Ξ are given as follows. For $\varepsilon = \pm 1$,

$$\begin{aligned}
&\chi_\theta \begin{pmatrix} \varepsilon \\ e^{-\delta\theta} \end{pmatrix} = -\varepsilon\delta \begin{pmatrix} \varepsilon \\ e^{-\delta\theta} \end{pmatrix} \quad \text{and} \\
&\Xi \begin{pmatrix} \varepsilon \\ e^{-\delta\theta} \end{pmatrix} = \begin{pmatrix} -\varepsilon\zeta_1 + \lambda e^{-\delta\theta} \\ -\varepsilon\lambda + \zeta_1 e^{-\delta\theta} \end{pmatrix} .
\end{aligned}$$

Thus if χ_θ and Ξ have any common eigenvectors, the vectors

$$\begin{pmatrix} \varepsilon \\ e^{-\delta\theta} \end{pmatrix} \text{ and } \begin{pmatrix} -\varepsilon\zeta_1 + \lambda e^{-\delta\theta} \\ -\varepsilon\lambda + \zeta_1 e^{-\delta\theta} \end{pmatrix}$$

must be multiples of each other. This implies

$$\det\begin{pmatrix}\varepsilon & -\varepsilon\zeta_1+\lambda e^{-\delta\theta}\\ e^{-\delta\theta} & -\varepsilon\lambda+\zeta_1 e^{-\delta\theta}\end{pmatrix}=2\varepsilon\zeta_1 e^{-\delta\theta}-\lambda(e^{-2\delta\theta}+\varepsilon^2)=0\,.$$

If $\zeta_1=0$, then $\lambda=0$ which is false. If $\zeta_1\neq 0$, then $0\neq\lambda\in\mathbb{R}$ which again is false. Assertion (1) now follows; Assertion (2) follows from Assertion (1) by Lemma 1.4.11.

Choose a compatible connection and express $A=\gamma_i\nabla_{e_i}+\psi_A$. Note that

$$\gamma_m\chi_\theta=-\chi_{-\theta}\gamma_m\quad\text{so}\quad\gamma_m\Pi_\theta^\pm=\Pi_{-\theta}^\mp\gamma_m\,.$$

We use Lemma 1.4.14 to see

$$\begin{aligned}&\int_M\Big\{\langle A\phi,\rho\rangle-\langle\phi,\tilde A\rho\rangle\Big\}dx=-\int_{\partial M}\langle\gamma_m\phi,\rho\rangle dy\\ =&\ -\int_{\partial M}\Big\{\langle\gamma_m\Pi_\theta^+\phi,\rho\rangle+\langle\gamma_m\Pi_\theta^-\phi,\rho\rangle\Big\}dy\\ =&\ -\int_{\partial M}\Big\{\langle\Pi_\theta^+\phi,\tilde\gamma_m\rho\rangle+\langle\gamma_m\phi,\tilde\Pi_{-\theta}^+\rho\rangle\Big\}dy\,.\end{aligned}$$

The same arguments as those given previously show that the adjoint boundary condition is $\tilde\Pi_{-\theta}^+$. □

1.6.6 Spectral boundary conditions

Absolute and relative boundary conditions arise in the study of the de Rham complex for manifolds with boundary. Spectral boundary conditions were introduced by Atiyah, Patodi, and Singer [9, 10, 11] in their study of the signature and spin complexes as these complexes do not admit local boundary conditions. This is closely related to the obstruction described in Lemma 1.6.6, with a suitable parity shift, that we shall discuss presently.

Let M be a compact Riemannian manifold with smooth boundary ∂M. Let γ be a Clifford module structure on a bundle V over M and let ∇ be a compatible connection on V. Let

$$P=\gamma_i\nabla_{e_i}+\psi_P$$

be an operator of Dirac type on a bundle V. In our discussion of bag boundary conditions in Section 1.6.5 for m even, we introduced an auxiliary endomorphism γ_{m+1}. We now assume $m=2\bar m+1$ is odd and define

$$\gamma_{m+1}:=(\sqrt{-1})^{\bar m+1}\gamma_1...\gamma_m\,.$$

The normalizing factor of $(\sqrt{-1})^{\bar m+1}$ is chosen so that $\gamma_{m+1}\gamma_{m+1}=\mathrm{Id}_V$. In contrast to the even dimensional setting, however, we now have that γ_{m+1} commutes with the γ_i for $1\le i\le m$. If M is orientable, then γ_{m+1} is globally defined. The fact that γ_{m+1} is only locally defined if M is not orientable will play no role in our discussion.

There is a *topological obstruction to the existence of elliptic local boundary conditions* in the odd dimensional setting:

Lemma 1.6.6 *Let P be an operator of Dirac type on a bundle V over an odd dimensional manifold M. If $\operatorname{Tr}(\gamma_{m+1}) \neq 0$, then there exists no bundle map B so (P, B) is elliptic with respect to the cone $\mathcal{K}$.*

Proof: Suppose to the contrary that (P, B) is elliptic with respect to the cone $\mathcal{K}$. We adopt the notation of Equation (1.4.i). Fix a point $y \in \partial M$. Let $u = (u_1, ..., u_m) \in \mathbb{R}^m$. The critical ellipticity condition comes for $\lambda \in \mathcal{K}$ purely imaginary. We therefore replace λ by $\sqrt{-1}u_m$ and ζ_a by u_a to define

$$\Xi(u) := \sqrt{-1}\gamma_m\gamma_a u_a - \sqrt{-1}\gamma_m u_m\,.$$

We then have $\Xi(u)^2 = |u|^2\operatorname{Id}_V$. Let $V_\pm(u)$ be the ± 1 eigenbundles of Ξ over the unit sphere S^{m-1} in $\mathbb{R}^m$. The boundary operator B is a linear map from $V|_{\partial M}$ to $\mathcal{W}$. Since (P, B) is elliptic with respect to the cone $\mathcal{K}$, by Lemma 1.4.9, B is an isomorphism from V_+ to W. Thus, in particular, restricting to the point $y \in \partial M$, we may conclude that the vector bundle V_+ is trivial over S^{m-1}.

Let $ch_{\bar m}$ be the $\bar m^{\text{th}}$ Chern character. We then have, see for example [189] (Lemma 2.1.5), that

$$\int_{S^{m-1}} ch_{\bar m}(V_+) = 2^{-\bar m}\operatorname{Tr}(\gamma_{m+1})\,.$$

Since V_+ is a trivial vector bundle, its characteristic classes vanish. This implies $\operatorname{Tr}(\gamma_{m+1}) = 0$, contrary to the hypothesis of the theorem. □

Since P need not admit local boundary conditions, it is natural to consider non-local boundary conditions. The matrices $\{-\gamma_m\gamma_1, ..., -\gamma_m\gamma_{m-1}\}$ define a $\operatorname{Clif}(T^*M)$ module on $V|_{\partial M}$ since

$$\gamma_m\gamma_a\gamma_m\gamma_b + \gamma_m\gamma_b\gamma_m\gamma_a = -\gamma_m\gamma_m(\gamma_a\gamma_b + \gamma_b\gamma_a) = -2g^{ab}\operatorname{Id}_V\,.$$

We say that an auxiliary operator A of Dirac type on $V|_{\partial M}$ is *admissible with respect to P* if $A = -\gamma_m\gamma_a\nabla_{e_a} + \psi_A$, where ψ_A is a suitably chosen 0^{th} order term, if there exists a fiber metric $(\cdot,\cdot)$ on $V|_{\partial M}$ so that the operator A is self-adjoint with respect to this metric, and if $\ker(A) = \{0\}$. We then let Π_A^+ be orthogonal projection on the span of the eigenspaces of A corresponding to **positive** eigenvalues. We introduce some auxiliary notation. In what follows ψ_* will always be a 0^{th} order operator. We have

$$\begin{aligned}
P &= \gamma_a\nabla_{e_a} + \gamma_m\nabla_{e_m} + \psi_P,\\
\tilde P &:= -\tilde\gamma_a\tilde\nabla_{e_a} - \tilde\gamma_m\tilde\nabla_{e_m} + \psi_{\tilde P},\\
A &= -\gamma_m\gamma_a\nabla_{e_a} + \psi_A,\\
\tilde A &:= -\tilde\gamma_m\tilde\gamma_a\tilde\nabla_{e_a} + \psi_{\tilde A},\\
A^\# &:= \tilde\gamma_m\tilde A\tilde\gamma_m = -\tilde\gamma_m\tilde\gamma_a\tilde\nabla_{e_a} + \psi_{A^\#}\,.
\end{aligned}$$

Thus $A^\#$ is admissible with respect to $\tilde P$ and defines a structure of the same type on the dual bundle V^*.

Lemma 1.6.7 *Let P be an operator of Dirac type on a vector bundle V over a compact Riemannian manifold M with smooth boundary ∂M. Let A be admissible with respect to P.*

1. *(P, Π_A^+) is elliptic with respect to the cone $\mathcal{K}$.*
2. *$\Pi_{A\#}^+$ defines the adjoint boundary condition for $\tilde{P}$.*
3. *We have the Green's formula*

$$\begin{aligned} &\langle P^2\phi, \rho\rangle_{L^2} - \langle \phi, \tilde{P}^2\rho\rangle_{L^2} \\ = &-\int_{\partial M} \Big\{ \langle \gamma_m \Pi_A^+ P\phi, \rho\rangle + \langle P\phi, \tilde{\gamma}_m \Pi_{A\#}^+ \rho\rangle + \langle \phi, \tilde{\gamma}_m \Pi_{A\#}^+ \tilde{P}\rho\rangle \\ &\qquad + \langle \gamma_m \Pi_A^+ \phi, \tilde{P}\rho\rangle \Big\}\, dy\,. \end{aligned}$$

4. *If $\gamma_m A = -A\gamma_m$ and if V admits a fiber metric so that P is formally self-adjoint, then the realization of P with respect to the boundary condition defined by Π_A^+ is self-adjoint.*
5. *We have $\psi_{\tilde{P}} = \tilde{\psi}_P$, $\psi_{\tilde{A}} = \tilde{\psi}_A$, and $\psi_{A\#} = \tilde{\gamma}_m \tilde{\psi}_A \tilde{\gamma}_m + L_{aa}\mathrm{Id}$.*

Proof: Let $V_\pm(\zeta, \lambda)$ be the eigenspaces of the operator

$$\Xi(\zeta, \lambda) := \sqrt{-1}\gamma_m \gamma_a \zeta_a - \gamma_m \lambda\,.$$

The argument given to prove Lemma 1.4.9 showed that if B was a bundle map, then (P, B) is elliptic with respect to the cone $\mathcal{K}$ if and only if

$$B : V_-(\zeta, \lambda) \xrightarrow{\approx} \mathcal{W} \quad \text{for all} \quad (0,0) \neq (\zeta, \lambda) \in T^*(\partial M) \times \mathcal{K},$$

the corresponding condition for V_+ was then deduced by replacing (ζ, λ) by $(-\zeta, -\lambda)$. Instead of considering a bundle map B, we are now considering a 0^{th} order pseudo-differential operator Π. Thus the relevant condition now is the requirement that $\sigma_L(\Pi_A^+)(\zeta)$ is an isomorphism of $V_-(\zeta, \lambda)$. We must assume that $\zeta \neq 0$ to ensure $\sigma_L(\Pi_A^+)(\zeta)$ is well defined. This is a crucial point that introduces additional technical difficulties in the analysis – see the discussion by Grubb [231].

By assumption,

$$\sigma_L(A)(\zeta) = -\sqrt{-1}\gamma_m \gamma_a \zeta_a = -\Xi(\zeta, 0)\,.$$

Thus the leading symbol of the pseudo-differential operator Π_A^+ is projection on the bundle V_- and hence, being the identity map, it is an isomorphism. Thus

$$\ker \sigma_L(\Pi_A^+)(\zeta) = V_+(\zeta, 0)\,.$$

The desired ellipticity now follows since

$$V_+(\zeta, 0) \cap V_-(\zeta, \lambda) = \{0\}\,.$$

Note that projection on the **negative** spectrum of A would **not** be elliptic with respect to the cone $\mathcal{K}$.

To prove the second assertion, we derive the corresponding Green's formula. By Lemma 1.4.14,

$$\begin{aligned}
&\int_M \left\{\langle P\phi,\rho\rangle - \langle\phi,\tilde P\rho\rangle\right\}dx = -\int_{\partial M}\langle\gamma_m\phi,\rho\rangle dy \\
= &\quad -\int_{\partial M}\left\{\langle\gamma_m\Pi_A^+\phi,\rho\rangle + \langle\gamma_m(\mathrm{Id}_V - \Pi_A^+)\phi,\rho\rangle\right\}dy \\
= &\quad -\int_{\partial M}\left\{\langle\gamma_m\Pi_A^+\phi,\rho\rangle + \langle\phi,(\mathrm{Id}_{V^*} - \tilde\Pi_A^+)\tilde\gamma_m\rho\rangle\right\}dy\,.
\end{aligned}$$

If γ is a suitably chosen path about the positive real axis, then

$$\Pi_A^+ = \tfrac{1}{2\pi\sqrt{-1}}\int_\gamma (A-\lambda)^{-1}d\lambda \quad\text{so}\quad \Pi_{\tilde A}^+ = \tfrac{1}{2\pi\sqrt{-1}}\int_\gamma(\tilde A-\lambda)^{-1}d\lambda = \tilde\Pi_A^+\,.$$

Consequently $\mathrm{Id}_{V^*} - \tilde\Pi_A^+$ is spectral projection on the negative eigenspaces of $\tilde A$. Conjugating with $\tilde\gamma_m$ yields that

$$(\tilde\gamma_m)^{-1}(\mathrm{Id}_{V^*} - \tilde\Pi_A^+)\tilde\gamma_m$$

is spectral projection on the negative eigenspaces of

$$(\tilde\gamma_m)^{-1}\tilde A\tilde\gamma_m = -\tilde\gamma_m\tilde A\tilde\gamma_m = -A^\#\,.$$

Consequently we have $(\tilde\gamma_m)^{-1}(\mathrm{Id}_{V^*} - \tilde\Pi_A^+)\tilde\gamma_m = \Pi_{A\#}^+$ so

$$(\mathrm{Id}_{V^*} - \tilde\Pi_A^+)\tilde\gamma_m = \tilde\gamma_m\Pi_{A\#}^+\,. \tag{1.6.k}$$

The second assertion now follows as we have

$$\begin{aligned}
&\int_M\left\{\langle P\phi,\rho\rangle - \langle\phi,\tilde P\rho\rangle\right\}dx \qquad (1.6.\mathrm{l})\\
&= -\int_{\partial M}\left\{\langle\gamma_m\Pi_A^+\phi,\rho\rangle + \langle\phi,\tilde\gamma_m\Pi_{A\#}^+\rho\rangle\right\}dy\,.
\end{aligned}$$

We replace ϕ by $P\phi$ in Equation (1.6.l) to see

$$\begin{aligned}
&\int_M\left\{\langle P^2\phi,\rho\rangle - \langle P\phi,\tilde P\rho\rangle\right\}dx \qquad (1.6.\mathrm{m})\\
&= -\int_{\partial M}\left\{\langle\gamma_m\Pi_A^+P\phi,\rho\rangle + \langle P\phi,\tilde\gamma_m\Pi_{A\#}^+\rho\rangle\right\}dy\,.
\end{aligned}$$

Interchanging the roles of ρ and ϕ means replacing γ_m by $-\tilde\gamma_m$. Therefore Equation (1.6.m) implies

$$\begin{aligned}
&\int_M\left\{\langle\phi,\tilde P^2\rho\rangle - \langle P\phi,\tilde P\rho\rangle\right\}dx \qquad (1.6.\mathrm{n})\\
= &\quad -\int_{\partial M}\left\{-\langle\phi,\tilde\gamma_m\Pi_{A\#}^+\tilde P\rho\rangle - \langle\gamma_m\Pi_A^+\phi,\tilde P\rho\rangle\right\}dy\,.
\end{aligned}$$

We may now derive Assertion (3) from Equations (1.6.m) and (1.6.n).

If $\gamma_m A = -A\gamma_m$, then $A^\# = \tilde A$; Assertion (4) follows. We apply Lemma

1.4.14 to see that the formal adjoint of P over M is given by $-\gamma_i\tilde{\nabla}_{e_i}+\tilde{\psi}_P$; thus $\psi_{\tilde{P}}=\tilde{\psi}_P$. We apply Lemma 1.4.15 to see

$$-\tilde{\gamma}_m\tilde{\gamma}_a\tilde{\nabla}_{e_a}+\tilde{\psi}_A$$

is the adjoint of A over ∂M. Consequently $\tilde{\psi}_A=\psi_{\tilde{A}}$. Finally, we compute

$$\begin{aligned}
A^{\#} &= \tilde{\gamma}_m\tilde{A}\tilde{\gamma}_m=-\tilde{\gamma}_m\tilde{\gamma}_m\tilde{\gamma}_a\tilde{\nabla}_{e_a}\tilde{\gamma}_m+\tilde{\gamma}_m\tilde{\psi}_A\tilde{\gamma}_m\\
&= \tilde{\gamma}_a\tilde{\gamma}_m\tilde{\nabla}_{e_a}+\tilde{\gamma}_a\Gamma_{amc}\tilde{\gamma}_c+\tilde{\gamma}_m\tilde{\psi}_A\tilde{\gamma}_m\\
&= -\tilde{\gamma}_m\tilde{\gamma}_a\tilde{\nabla}_{e_a}-\tilde{\gamma}_a\tilde{\gamma}_cL_{ac}+\tilde{\gamma}_m\tilde{\psi}_A\tilde{\gamma}_m\\
&= -\tilde{\gamma}_m\tilde{\gamma}_a\tilde{\nabla}_{e_a}+L_{aa}\mathrm{Id}+\tilde{\gamma}_m\tilde{\psi}_A\tilde{\gamma}_m\,.
\end{aligned}$$

This completes the proof of the final assertion. $\square$

We have assumed that A is self-adjoint and that $\ker(A)=\{0\}$ only to simplify the exposition. Even if A is not self-adjoint, an appropriate generalization of Theorem 1.3.4 shows the spectrum is contained near the x axis. Consequently in the general setting, one could let Π_A^+ be the spectral projection on the generalized eigenspaces corresponding to $\Re(\lambda)>0$; the adjoint boundary condition $\Pi_{A^{\#}}^+$ would then be the spectral projection on the eigenspaces of $A^{\#}$ corresponding to $\Re(\lambda)\geq 0$. We emphasize that $\tilde{A}$ does not necessarily define the adjoint boundary condition on V^*.

1.6.7 Non-minimal operators

Let A and B be positive constants. Let

$$D=Ad\delta+B\delta d+E\quad\text{on}\quad C^\infty(\Lambda^p(M))\,.$$

This operator is said to be a *non-minimal operator* in the physics literature [83, 239]. It is not an operator of Laplace type for $A\neq B$. Let $\mathfrak{e}$ and $\mathfrak{i}$ be exterior and interior multiplication. We then have

$$\sigma_L(D)=A\mathfrak{e}(\xi)\mathfrak{i}(\xi)+B\mathfrak{i}(\xi)\mathfrak{e}(\xi)\,.$$

Thus the leading symbol of D is self-adjoint. Let $0\neq\xi$. We choose an orthonormal basis $\{e_i\}$ so $\xi=ce_1$. Let $1\leq i_1<...<i_p\leq m$. By Equation (1.2.e)

$$\sigma_L(D)(\zeta)\{e_{i_1}\wedge...\wedge e_{i_p}\}=c^2\begin{cases}Ae_{i_1}\wedge...\wedge e_{i_p} & \text{if}\quad i_1=1,\\ Be_{i_1}\wedge...\wedge e_{i_p} & \text{if}\quad i_1>1\,.\end{cases}$$

This shows that the symbolic spectrum of D is $(0,\infty)$ and thus D is elliptic with respect to the cone $\mathcal{C}$. With a bit more work, the same arguments as those used in the discussion of Section 1.5 can be used to show that $(D,\mathcal{B})$ is elliptic with respect to the cone $\mathbb{C}$ when $\mathcal{B}$ denotes absolute or relative boundary conditions. We omit details in the interest of brevity.

Let D be an operator of Laplace type on a vector bundle V and let $\mathcal{B}_T$ be a first order tangential partial differential operator on $V|_{\partial M}$. We follow the discussion in [153, 265, 350] to define the *oblique boundary operator* by setting

$$\mathcal{B}\phi := \phi_{;m}|_{\partial M} + \mathcal{B}_T(\phi|_{\partial M})\,.$$

We consider a slightly more general notion of ellipticity. Let

$$\mathcal{C}_\delta := \{z = r(\cos\theta + \sqrt{-1}\sin\theta) \in \mathbb{C} : \theta \in (\delta, 2\pi - \delta),\ r \geq 0\}$$

be the complement of an open cone of angle δ about the positive real axis.

Lemma 1.6.8 *Let D be an operator of Laplace type on a vector bundle V over compact Riemannian manifold with smooth boundary. Let $0 < \delta < \frac{\pi}{2}$. There exists $\varepsilon = \varepsilon(\delta, D)$ so that if $\mathcal{B}_T$ is any tangential first order operator on $V|_{\partial M}$ with $|\sigma_L(\mathcal{B}_T)(\zeta)| \leq \varepsilon|\zeta|$ for all $\zeta \in T^*M$, then:*

1. *$(D, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}_\delta$.*
2. *$\tilde{\mathcal{B}} := (\tilde\nabla_{e_m} + \tilde{\mathcal{B}}_T)$ defines the adjoint boundary condition for $\tilde D$.*
3. *$e^{-tD_{\mathcal{B}}}$ is well defined for $t > 0$.*

Proof: By Lemma 1.3.1, D is elliptic with respect to the cone $\mathcal{C}$ and hence with respect to the smaller cone $\mathcal{C}_\delta$. Let

$$\mathcal{B}_T = \Gamma_a \nabla_{e_a} + S\,.$$

The operator of Equation (1.4.h) is then given by

$$\mathfrak{b}(\zeta, \lambda) := \sqrt{-1}\cdot\Gamma_a\zeta_a - \sqrt{|\zeta|^2 - \lambda}\cdot \mathrm{Id}_V\,.$$

To establish Assertion (1), we must show that $\mathfrak{b}$ is invertible for

$$(0,0) \neq (\zeta, \lambda) \in T^*\partial M \times \mathcal{C}_\delta\,.$$

Since $\mathfrak{b}(c\zeta, c^2\lambda) = c\mathfrak{b}(\zeta, \lambda)$, we can restrict to the compact domain

$$\mathcal{D} := \{(\zeta, \lambda) \in T^*(\partial M) \times \mathbb{C} : |\zeta|^2 + |\lambda| = 1 \text{ and } \arg(\lambda) \in [\delta, 2\pi - \delta]\}\,.$$

Since $\mathfrak{b}$ is invertible when $\Gamma = 0$ and since $\mathcal{D}$ is compact, $\mathfrak{b}$ is invertible for small Γ by continuity. Assertion (1) now follows – it was to ensure this compactness that we needed to introduce the additional parameter δ.

We determine the associated *Green's formula* to prove the second assertion. By Lemma 1.4.17,

$$\begin{aligned}
&\int_M \left\{\langle D\phi, \rho\rangle - \langle \phi, \tilde D\rho\rangle\right\}dx = \int_{\partial M}\left\{\langle \phi_{;m}, \rho\rangle - \langle\phi, \rho_{;m}\rangle\right\}dy \\
&= \int_{\partial M}\left\{\langle \phi_{;m} + \mathcal{B}_T\phi, \rho\rangle - \langle \phi, \rho_{;m} + \tilde{\mathcal{B}}_T\rho\rangle\right\}dy \qquad (1.6.\mathrm{o}) \\
&= \int_{\partial M}\left\{\langle \mathcal{B}\phi, \rho\rangle - \langle\phi, \tilde{\mathcal{B}}\rho\rangle\right\}dy\,.
\end{aligned}$$

Assertion (2) now follows from Lemma 1.4.1 and Definition 1.4.12.

As Theorem 1.3.4 generalizes to this setting, the spectrum of $D_{\mathcal{B}}$ is contained in the region $\mathcal{R}_{\delta_1,n}$ defined in Equation (1.3.a) where $0 < \delta < \delta_1 < \frac{\pi}{4}$ and $n = n(D, \mathcal{B}_T, \delta_1)$. The final assertion now follows. $\square$

1.6.9 Time-dependent heat conduction problems

Let D_0 be an initial operator of Laplace type on V. Let ∇ be the connection defined by D_0 on V using Lemma 1.2.1.

Let $\mathfrak{D} := \{D_t\}_{t\geq 0}$ be a smooth 1 parameter family of time-dependent operators of Laplace type. Let $\mathfrak{B} := \{\mathcal{B}_t\}_{t\geq 0}$ be a smooth 1 parameter family of time-dependent boundary conditions. We suppose $(D_t, \mathcal{B}_t)$ is elliptic with respect to a cone $\mathcal{C}_\delta$ for some $0 \leq \delta < \frac{\pi}{2}$ and for all t in the parameter range. If ϕ is the initial temperature distribution, then the subsequent temperature distribution $u(x;t)$ is characterized by the equations:

$$\begin{aligned}
&(\partial_t + D_t)u(x;t) = 0 && \text{for} \quad t > 0 && \text{(evolution equation)},\\
&\mathcal{B}_t u(\cdot;t) = 0 && \text{for} \quad t > 0 && \text{(boundary condition)},\\
&u|_{t=0} = \phi && && \text{(initial condition)}\,.
\end{aligned} \tag{1.6.p}$$

The final equality is to be understood as meaning that $\lim_{t\downarrow 0} u(\cdot,t) = \phi$ in the distributional sense. We shall formally let $u = e^{-t\mathfrak{D}_{\mathcal{B}}}\phi$ denote the solution to Equation (1.6.p). This is **not** defined by the functional calculus.

Let ρ be the specific heat. Let dx_0 be the Riemannian metric of the initial background metric g_0. The heat content

$$\beta(\phi, \rho, \mathfrak{D}, \mathfrak{B})(t) := \int_M \langle u(x;t), \rho(x)\rangle dx_0$$

is then well defined and, as $t \downarrow 0$, there is a complete asymptotic expansion

$$\beta(\phi, \rho, \mathfrak{D}, \mathfrak{B})(t) \sim \sum_{n=0}^{\infty} \beta_n(\phi, \rho, \mathfrak{D}, \mathfrak{B}) t^{n/2}\,.$$

More generally, of course, one could consider specific heats $\rho(x;t)$ which were time-dependent. Expanding ρ in a Taylor series

$$\rho(x;t) \sim \sum_{r=0}^{\infty} \rho_r(x) t^r$$

would then lead to an asymptotic expansion

$$\begin{aligned}
\beta(\phi, \rho, \mathfrak{D}, \mathfrak{B})(t) &\sim \sum_{r=0}^{\infty} \beta(\phi, \rho_r, \mathfrak{D}, \mathfrak{B})(t) \cdot t^r\\
&\sim \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \beta_n(\phi, \rho_r, \mathfrak{D}, \mathfrak{B}) t^{r+\frac{1}{2}n}
\end{aligned}$$

and no new information would result. Consequently, we shall assume henceforth that ρ is *static*; this means that ρ is independent of the time parameter.

We must express $\mathfrak{D}$ tensorially. Let $\{e_1, ..., e_m\}$ be a local frame field for TM which is orthonormal with respect to the initial metric g_0. We may then define scalar symmetric 2 tensor fields $\mathcal{G}_{r,ij} = \mathcal{G}_{r,ji}$, endomorphism valued 1-tensor fields $\mathcal{F}_{r,i}$, and endomorphism valued tensor fields $\mathcal{E}_r$ by expanding D_t in a Taylor series

$$D_t\phi \sim D_0\phi + \sum_{r=1}^{\infty} \left\{ \mathcal{G}_{r,ij}\phi_{;ij} + \mathcal{F}_{r,i}\phi_{;i} + \mathcal{E}_r\phi \right\} t^r .$$

Let $\{e^1, ..., e^m\}$ be the dual frame for T^*M. Since $g_t(\xi, \xi) = -\sigma_L(D_t)(\xi)$,

$$g_t(e^i, e^j) \sim \delta^{ij} - \sum_{r=1}^{\infty} \mathcal{G}_{r,ij} t^r .$$

Thus, dually, the metric on TM has the form

$$g_t(e_i, e_j) = \delta_{ij} + \mathcal{G}_{1,ij} t + O(t^2),$$

which justifies the sign convention that we have chosen.

Since our study is motivated by the ordinary heat equation in the context of variable geometries, we must express the scalar Laplacians for a family of time-dependent metrics in this setting. It suffices to compute $\mathcal{G}_1$, $\mathcal{F}_1$, and $\mathcal{E}_1$ since the higher order terms will play no role in the computation of the heat content asymptotics β_n or the heat trace asymptotics a_n for $n \leq 4$ as we shall see subsequently.

Lemma 1.6.9 *Let $g_t = g_0 + ht + O(t^2)$ be a smooth 1 parameter family of Riemannian metrics on M. Let D_t be the associated scalar Laplacians. Then*

$$\mathcal{G}_1 = h, \quad \mathcal{F}_{1,i} = h_{ij;j} - \tfrac{1}{2} h_{jj;i}, \quad \textit{and} \quad \mathcal{E}_1 = 0 .$$

Proof: We follow the discussion in [190] to prove this Lemma. We correct a minor mistake in the computation which was given there. Let

$$G_{\mu\nu}(t) = g_{\mu\nu} + th_{\mu\nu} + O(t^2) \quad \text{and} \quad G := \sqrt{\det G_{\mu\nu}} .$$

Then the Laplacian is given by

$$\Delta_t = -G^{-1} \partial_\mu G^{\mu\nu} G \partial_\nu .$$

Fix a point $P \in M$ and choose local coordinates $x = (x_1, ..., x_m)$ centered at P so $g_{\mu\nu}(P) = \delta_{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}(P) = 0$. We then have

$$\begin{aligned}
&g_{\mu\nu} = \delta_{\mu\nu} + th_{\mu\nu} + O(|x|^2) + O(t^2), \\
&G = 1 + \tfrac{1}{2} th_{\sigma\sigma} + O(|x|^2) + O(t^2), \\
&G^{\mu\nu} = \delta_{\mu\nu} - th_{\mu\nu} + O(|x|^2) + O(t^2), \\
&\partial_\mu \partial_\nu \phi = \phi_{;\mu\nu} + O(|x|), \\
&\partial_\mu h_{\mu\nu} - \tfrac{1}{2} \partial_\nu h_{\mu\mu} = h_{\mu\nu;\mu} - \tfrac{1}{2} h_{\mu\mu;\nu} + O(|x|) .
\end{aligned}$$

Let $h_{\nu\sigma/\mu} := \partial_\mu h_{\nu\sigma}$. We may now compute:

$$\begin{aligned}
\Delta_t\phi &= -\{(1-\tfrac{1}{2}th_{\varrho\varrho})\partial_\mu(\delta_{\mu\nu}-th_{\mu\nu})(1+\tfrac{1}{2}th_{\varepsilon\varepsilon})\partial_\nu\}\phi + O(|x|) + O(t^2) \\
&= \{\Delta_0 + t(h_{\mu\nu}\partial_\mu\partial_\nu + (h_{\mu\nu/\mu} - \tfrac{1}{2}h_{\mu\mu/\nu})\partial_\nu)\}\phi + O(|x|) + O(t^2) \\
&= \Delta_0\phi + t(h_{\mu\nu}\phi_{;\mu\nu} + (h_{\mu\nu;\mu} - \tfrac{1}{2}h_{\mu\mu;\nu})\phi_{;\nu}) + O(|x|) + O(t^2)\,.
\end{aligned}$$

Evaluating at $x = 0$ then yields

$$\begin{aligned}
&\mathcal{G}_{1,\mu\nu}(P) = h_{\mu\nu}(P), \\
&\mathcal{F}_{1,\nu}(P) = (h_{\mu\nu;\mu} - \tfrac{1}{2}h_{\mu\mu;\nu})(P), \\
&\mathcal{E}_1(P) = 0\,.
\end{aligned}$$

The Lemma now follows as the point P was arbitrary. $\square$

1.7 Invariance theory

1.7.1 The first and second theorems of invariance theory

Let V be an m dimensional real vector space, let $g(\cdot,\cdot)$ be a positive definite symmetric inner product on V, and let $O(V)$ be the associated orthogonal group, defined by

$$O(V) := \{\xi \in GL(V) : g(\xi v, \xi v) = g(v,v) \forall v \in V\}\,.$$

By choosing an orthonormal basis for V, we could identify V with $\mathbb{R}^m$ and thereby identify $O(V)$ with $O(m)$. We shall not do this as the whole point is to work in a basis free fashion and thereby derive results which can be used to determine local invariants of a Riemannian manifold.

Following Weyl, we shall say that a polynomial map

$$f : \times^k V \to \mathbb{R}$$

is an *orthogonal invariant* if the following identity holds for all ξ in $O(V)$ and for all $(v_1,...,v_k)$ in $\times^k V$

$$f(\xi v_1,...,\xi v_k) = f(v_1,...,v_k)\,.$$

Let $\mathcal{A}_{k,V}$ be the commutative unital algebra of all such orthogonal invariants. Introduce elements

$$A_{ij} = A_{ji} := g(v_i,v_j) \in \mathcal{A}_{k,V}\,.$$

We then have by Weyl [361]:

Theorem 1.7.1 (First main theorem of invariants) *Every orthogonal invariant which depends on k vectors $(v_1,...,v_k)$ in $\times^k V$ is expressible in terms of the k^2 scalar invariants $g(v_i,v_j)$.*

The algebra $\mathcal{A}_{k,V}$ is not a free algebra in the variables $\{A_{ij}\}$. Let

$$\begin{aligned}\mathfrak{R}_k(v_1,...,v_k;w_1,...,w_k) : &= \det\begin{pmatrix} g(v_1,w_1) & ... & g(v_k,w_1) \\ ... & ... & ... \\ g(v_1,w_k) & ... & g(v_k,w_k)\end{pmatrix} \quad (1.7.a)\\ &= g(v_1\wedge...\wedge v_{m+1},w_1\wedge...\wedge w_{m+1}).\end{aligned}$$

We also have by Weyl [361]:

Theorem 1.7.2 (Second main theorem of invariants) *If* $\dim(V)=m$, *then every relation among scalar products is an algebraic consequence of the relations* $\mathfrak{R}_{2m+2}(v_1,...,v_{m+1};w_1,...,w_{m+1})=0$.

These two results can be given a slightly more algebraic flavor as follows. We consider the free polynomial algebra $\tilde{\mathcal{A}}_{k,V} := \mathbb{R}[A_{ij}]$ in $\frac{1}{2}k(k+1)$ formal variables $A_{ij}=A_{ji}$ for $1\le i,j\le k$. The evaluation

$$e(A_{ij})(v_1,...,v_k) := g(v_i,v_j)$$

induces a natural algebra homomorphism

$$e:\tilde{\mathcal{A}}_{k,V}\rightarrow\mathcal{A}_{k,V}$$

which is surjective by Theorem 1.7.1. Theorem 1.7.2 identifies the kernel of the evaluation e with the ideal of $\tilde{\mathcal{A}}_{k,V}$ which is generated by the determinants described in Theorem 1.7.2.

Let $\mathcal{I}_{k,V}$ be the set of all multilinear invariant maps from $\times^kV$ to $\mathbb{R}$. Since we are considering the full orthogonal group and not the special orthogonal group, the map $\xi: v\rightarrow -v$ is permissible. If f is multi-linear, then

$$f(-v_1,...,-v_k)=(-1)^kf(v_1,...,v_k).$$

Thus there are no multilinear maps which are orthogonally invariant if k is odd so we assume k is even henceforth. We apply Theorem 1.7.1 to study this space.

Let Σ_k be the group of all permutations of the set of indices $\{1,...,k\}$. We define a multi-linear invariant map $p_{k,\sigma}$ for any permutation $\sigma\in\Sigma_k$ by setting

$$p_{k,\sigma}(v_1,...,v_k) := g(v_{\sigma(1)},v_{\sigma(2)})\cdots g(v_{\sigma(k-1)},v_{\sigma(k)}). \quad (1.7.b)$$

Theorem 1.7.3 $\mathcal{I}_{k,V}=\operatorname{Span}_{\sigma\in\Sigma_k}\{p_{k,\sigma}\}$.

Proof: Let $I=(i_1,...,i_{2\ell})$ for $1\le i_\mu\le m$ be a multi-index. Set

$$f_I(v_1,...,v_k) := g(v_{i_1},v_{i_2})...g(v_{i_{2\ell-1}},v_{i_{2\ell}})\in\mathcal{A}_{k,V}.$$

We may then use Theorem 1.7.1 to decompose $p\in\mathcal{I}_{k,V}$ in the form

$$p=\sum_I a_If_I; \quad (1.7.c)$$

in light of Theorem 1.7.2 we note that this decomposition need not be unique and thus we must proceed with a bit of caution.

Let $(x_1, ..., x_k) \in \mathbb{R}^k$. Since p is multi-linear,

$$p(x_1 v_1, ..., x_k v_k) = x_1 ... x_k \cdot p(v_1, ..., v_k)\,.$$

Let $\mathcal{D}_k := \partial_1^x ... \partial_k^x$. Since $p(v_1, ..., v_k) = \mathcal{D}_k p(x_1 v_1, ..., x_k v_k)|_{x=0}$, Equation (1.7.c) implies

$$p(v) = \sum_I a_I \mathcal{D}_k f_I(x_1 v_1, ..., x_k v_k)|_{x=0}\,. \tag{1.7.d}$$

For $1 \le i \le k$, let $n_I(i)$ be the number of times that the index i appears in I. We then have

$$f_I(x_1 v_1, ..., x_k v_k) = f_I(v_1, ..., v_k) \prod_{i=1}^m x_i^{n_I(i)}\,.$$

Consequently $\mathcal{D}_k f_I(x_1 v_1, ..., x_k v_k)|_{x=0} = 0$ if $n_I(i) \ne 1$ for any index i. Thus we may restrict the sum in Equations (1.7.c) and (1.7.d) to range over multi-indices I which are a permutation of the indices $\{1, ..., k\}$. The Lemma now follows as these are exactly the invariants $p_{k,\sigma}$ which were defined in Equation (1.7.b). $\square$

In view of Theorem 1.7.3, one says "invariant multilinear maps are given by contractions of indices" as, relative to an orthonormal basis, the inner products involved correspond to contraction of indices in pairs. Invariant multi-linear maps from $\times^k V$ to $\mathbb{R}$ are, of course, nothing but invariant linear maps from $\otimes^k V$ to $\mathbb{R}$. Let $\{e_1, ..., e_m\}$ be an orthonormal basis for the vector space V. Let $\omega = \omega_{ij} e^i \otimes e^j \in \otimes^2 V$. We then have

$$\mathcal{I}_{2,V} = \operatorname{Span}\{p_1\} \quad \text{where} \quad p_1 : \omega \to \omega_{ii}\,.$$

If $\omega = \omega_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l \in \otimes^4 V$, then set

$$p_2 : \omega \to \omega_{iijj},\; p_3 : \omega \to \omega_{ijij},\; p_4 : \omega \to \omega_{ijji}\,.$$

One then has

$$\mathcal{I}_{4,V} = \operatorname{Span}\{p_2, p_3, p_4\}\,. \tag{1.7.e}$$

Let W be a vector space of dimension $m-1$. Choose an inner product preserving inclusion $i : W \subset V$ and a corresponding embedding of the orthogonal group $O(W) \subset O(V)$. We define a *restriction map*

$$\mathfrak{R} : \mathcal{I}_{k,V} \to \mathcal{I}_{k,W}$$

which is characterized dually by the property

$$\mathfrak{R}(p)(w_1, ..., w_k) = p(i(w_1), ..., i(w_k))\,.$$

If $k \ge 2m$ and if $\sigma \in \Sigma_k$, define

$$\begin{aligned} r_{k,m,\sigma}(v_1, ..., v_k) :\; &= g(v_{\sigma(1)} \wedge ... \wedge v_{\sigma(m)}, v_{\sigma(m+1)} \wedge ... \wedge v_{\sigma(2m)}) \\ &\times g(v_{\sigma(2m+1)}, v_{\sigma(2m+2)}) \cdots g(v_{\sigma(k-1)}, v_{\sigma(k)})\,. \end{aligned}$$

We can also express

$$
\begin{aligned}
r_{k,m,\sigma} &= \det\begin{pmatrix} g(v_{\sigma(1)},v_{\sigma(m+1)}) & \cdots & g(v_{\sigma(m)},v_{\sigma(2m)}) \\ \cdots & \cdots & \cdots \\ g(v_{\sigma(1)},v_{\sigma(m+1)}) & \cdots & g(v_{\sigma(m)},v_{\sigma(2m)}) \end{pmatrix} \\
&\cdot \quad A_{\sigma(2m+1),\sigma(2m+2)}\cdots A_{\sigma(2k-1)\sigma(2k)}\,.
\end{aligned}
$$

Theorem 1.7.4 *Let $m \geq 2$.*

1. $\mathfrak{R} : \mathcal{I}_{k,V} \to \mathcal{I}_{k,W}$ *is surjective.*
2. $\mathfrak{R} : \mathcal{I}_{k,V} \to \mathcal{I}_{k,W}$ *is injective if $k < 2m$.*
3. *If $k \geq 2m$, then* $\ker(\mathfrak{R}) \cap \mathcal{I}_{k,V} = \operatorname{Span}_{\sigma\in\Sigma_k}\{r_{k,m,\sigma}\}$.

Proof: If p is given by contractions of indices which range from 1 to m, then $\mathfrak{R}(p)$ is given by restricting the range of summation to range from 1 to $m-1$. Consequently, by Theorem 1.7.3, the map $\mathfrak{R}$ is surjective.

To prove Assertion (2), we use Theorem 1.7.1 to express $p \in \mathcal{I}_{k,V}$ in terms of inner products. We use the second main theorem of invariants 1.7.2, after making an appropriate dimension shift, to see that $\mathfrak{R}(p)$ vanishes if and only if it can be written as sums of terms each of which is divisible by an appropriate determinant J of size $m \times m$. The desired result now follows from Equation (1.7.a) and from the same arguments used to prove Theorem 1.7.3. □

1.7.2 Local invariants of a Riemannian metric

We shall be interested in local invariants of Riemannian manifolds. In light of Theorem 1.1.1, any local invariant of the Riemannian metric which is polynomial in the jets of the derivatives of the metric with coefficients which are smooth functions of the metric can be expressed polynomially in terms of the curvature tensor. Such invariants can be constructed as follows. Let

$$P = P(R_{i_1i_2i_3i_4}, R_{i_1i_2i_3i_4;j_1}, ..., R_{i_1i_2i_3i_4;j_1,...,j_\ell})$$

be a polynomial formula in the components of the covariant derivatives of the Riemann curvature tensor. This is a formal object that can be evaluated on a Riemannian metric g once a local orthonormal frame e is chosen. The algebra of such formulae is not a free polynomial algebra as we must take into account the symmetries of the covariant derivatives curvature tensor; see, for example, the relations in Equations (1.1.b), (1.1.d), and (1.1.h). We say that P is invariant if the value of $P(g,e)$ is independent of the particular orthonormal frame e which is chosen and depends only on the underlying metric g. Let $\mathcal{P}_m$ be the space of all such invariants.

The space $\mathcal{P}_m$ inherits a natural grading. We define the *weight* of R_{ijkl} to be 2 and increase the weight by 1 for every explicit covariant derivative present;

$$\operatorname{weight}(R_{i_1i_2i_3i_4;j_1,...,j_\ell}) := 2 + \ell\,.$$

The notion of weight is well defined; the identities giving the curvature symmetries are weight homogeneous; see for example the relations in Equations (1.1.b), (1.1.d), and (1.1.h).

We can also apply *dimensional analysis* to see that the weight is well defined. Let $0 \neq c \in \mathbb{R}$. We rescale the Riemannian metric to define $g_c := c^{-2}g$ and rescale the orthonormal frame by defining $e_c := ce$. Since the Levi-Civita connection is unchanged, we may compute

$$\begin{aligned} R_{ijkl}(g_c, e_c): &= g_c(\{\nabla_{ce_i}\nabla_{ce_j} - \nabla_{ce_j}\nabla_{ce_i} - \nabla_{[ce_i, ce_j]}\}ce_k, ce_l) \\ &= c^4 \cdot c^{-2} g(\{\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i, e_j]}\}e_k, e_l) \\ &= c^2 R_{ijkl}(g, e)\,. \end{aligned}$$

More generally, a similar computation shows that

$$R_{i_1 i_2 i_3 i_4; j_1 \dots j_l}(g_c, e_c) = c^{2+\ell} R_{i_1 i_2 i_3 i_4; j_1 \dots j_l}(g, e)\,. \tag{1.7.f}$$

We use Equation (1.7.f) to decompose

$$\mathcal{P}_m = \oplus_n \mathcal{P}_{n,m}$$

where the elements of $\mathcal{P}_{n,m}$ are homogeneous of weight n. Note that if P belongs to $\mathcal{P}_{n,m}$, then P satisfies the rescaling relation

$$P(c^{-2}g) = c^n P(g)\,. \tag{1.7.g}$$

By taking $c = -1$, we see that $\mathcal{P}_{n,m} = \{0\}$ if n is odd.

Atiyah, Bott, and Patodi [7] used Weyl's first theorem of invariants (Theorem 1.7.1) to analyze these and related spaces of invariants. We follow their approach to show:

Lemma 1.7.5 *1.* $\mathcal{P}_{0,m} = \text{Span}\{1\}$.

2. $\mathcal{P}_{2,m} = \text{Span}\{\tau := R_{ijji}\}$.

3. $\mathcal{P}_{4,m} = \text{Span}\{\tau^2,\ |\rho^2| := R_{ijjk}R_{illk},\ |R|^2 := R_{ijkl}R_{ijkl},\ \Delta\tau := -R_{ijji;kk}\}$.

Proof: Let V be an m dimensional vector space which is equipped with a positive definite inner product $g(\cdot,\cdot)$. We say that a 4 tensor $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* if R satisfies the curvature symmetries of the Riemann curvature tensor given in Display (1.1.b).

Suppose that $n = 2$. Then $P \in \mathcal{P}_{2,m}$ is linear in the components of the curvature tensor. Let $W \subset \otimes^4 V^*$ be the subspace of all algebraic curvature tensors. We regard P as a map from W to $\mathbb{R}$ which is orthogonally invariant. Extending P to be zero on $W^\perp$ defines a multi-linear orthogonally invariant map to which Theorem 1.7.3 applies. Thus by Equation (1.7.e)

$$P(g) = c_1 R_{iijj} + c_2 R_{ijij} + c_3 R_{ijji}\,.$$

Assertion (2) now follows since, after taking into account the symmetries of Display (1.1.b), we have that

$$P(g) = (c_3 - c_2) R_{ijji}\,.$$

Next suppose that $n = 4$. We now have that $P \in \mathcal{P}_{4,m}$ is quadratic in R and linear in $\nabla^2 R$. Again, P can be regarded as a map from a certain subspace

$$W \subset \{\otimes^6 T^* M\} \oplus \{\otimes^8 T^* M\}$$

to $\mathbb{R}$ which is invariant under the action of the orthogonal group; here W is the subspace which is generated by

$$\nabla^2 R \oplus R \otimes R \subset \otimes^6 T^* M \oplus \otimes^8 T^* M\,.$$

The relation of Equation (1.1.h) now enters.

As W is orthogonally invariant, extending P to be zero on $W^\perp$ defines an orthogonally invariant map to which Theorem 1.7.3 applies and Assertion (3) follows after reducing by the appropriate curvature symmetries. □

More generally, a spanning set for $\mathcal{P}_{n,m}$ is given by contracting indices for the Riemann curvature tensor and its covariant derivatives and then reducing by the appropriate curvature symmetries.

On the boundary, the normal vector field plays a distinguished role and thus the structure group is $O(m-1)$ and not $O(m)$. Let $\tilde{\mathcal{P}}_{n,m}$ be the space of all boundary invariants in the derivatives of the metric which are homogeneous of weight n.

Let y be a system of geodesic polar coordinates centered at a point P of ∂M and let $x = (y, x_m)$ be the induced coordinates near the boundary as described in Section 1.1.2. We use Theorem 1.1.3 to see that any boundary invariant in the derivatives of the metric can be expressed in terms of the covariant derivatives of the curvature tensor $R = R_M$ and the tangential derivatives of the second fundamental form L. H. Weyl's theorem then applies. Reducing by the appropriate relations is a bit more complicated as relations of the form given in Lemma 1.1.4 must be taken into account. The following Lemma then follows:

Lemma 1.7.6 *1.* $\tilde{\mathcal{P}}_{0,m} = \operatorname{Span}\{1\}$.

2. $\tilde{\mathcal{P}}_{1,m} = \operatorname{Span}\{L_{aa}\}$.

3. $\tilde{\mathcal{P}}_{2,m} = \operatorname{Span}\{L_{aa}L_{bb}, L_{ab}L_{ab}, R_{ijji}, R_{amma}\}$.

In Chapter 2 and in Chapter 3, we will study more complicated spaces of invariants, but the basic approach will be similar. We refer, for example, to Lemmas 2.2.12 and 2.2.13 in the context of the heat content asymptotics and to Lemmas 3.1.10 and 3.1.11 in the context of the heat trace asymptotics.

1.7.3 The heat trace asymptotics for operators of Laplace type

Lemma 1.3.6 discusses the heat trace asymptotics for quite general elliptic operators whose symbolic spectrum is a subset of $(0,\infty)$. We can use Theorem 1.7.3 to simplify the discussion considerably for operators of Laplace type. Let $D = D(\nabla, E)$ be an operator of Laplace type. Let Ω be the curvature operator of the associated connection ∇; relative to a local frame field we have that

$$\Omega_{\mu\nu} : \quad = \quad \nabla_{\partial_\mu^x}\nabla_{\partial_\nu^x} - \nabla_{\partial_\nu^x}\nabla_{\partial_\mu^x}$$

$$= \quad \partial_\mu^x \omega_\nu - \partial_\nu^x \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu \,.$$

We introduce formal endomorphism valued invariants $\nabla^k E$ and $\nabla^k \Omega$. Set

$$\begin{aligned}
&\text{weight}\,(R_{i_1 i_2 i_3 i_4;j_1 \ldots j_\ell}) = 2 + \ell,\\
&\text{weight}\,(E_{;j_1 \ldots j_\ell}) = 2 + \ell,\\
&\text{weight}\,(\Omega_{i_1 i_2;j_1 \ldots j_\ell}) = 2 + \ell\,.
\end{aligned}$$

If we expand $\nabla^\ell R$, $\nabla^\ell E$, and $\nabla^\ell \Omega$ in terms of the jets of the total symbol of D, then we may use Lemma 1.2.1 to see that this definition of the weight is compatible with the definition given previously in Equation (1.3.b). We note that Equation (1.7.f) generalizes to this setting to yield

$$\begin{aligned}
&R_{i_1 i_2 i_3 i_4;j_1 \ldots j_\ell}(c^2 D, e_c) = c^{2+\ell} R_{i_1 i_2 i_3 i_4;j_1 \ldots j_\ell}(D, e),\\
&E_{;j_1 \ldots j_\ell}(c^2 D, e_c) = c^{2+\ell} E_{;j_1 \ldots j_\ell}(D, e),\\
&\Omega_{i_1 i_2;j_1 \ldots j_\ell}(c^2 D, e_c) = c^{2+\ell} \Omega_{i_1 i_2;j_1 \ldots j_\ell}(D, e)\,.
\end{aligned} \tag{1.7.h}$$

We have not introduced local coordinates on M nor have we introduced a local frame field $\vec{s}$ for V, preferring to work invariantly instead. The following observation provides a useful restatement of Lemma 1.3.6 in an invariant framework. As the proof is a straightforward application of the techniques we have discussed above, we shall omit the proof. We shall return to this question again in Section 3.1.

Lemma 1.7.7 *Let $D = D(\nabla, E)$ be an operator of Laplace type. Then the heat invariant $e_n(x, D)$ is given by a universal non-commutative polynomial of total weight n in the variables*

$$\{\nabla^k R, \nabla^k E, \nabla^k \Omega\}\,.$$

For example, it will follow from Theorem 3.3.1 that

$$\begin{aligned}
e_0(x, D) &= (4\pi)^{-m/2}\mathrm{Id}\,,\\
e_2(x, D) &= (4\pi)^{-m/2}\tfrac{1}{6}(6E + \tau \mathrm{Id}\,),\\
e_4(x, D) &= (4\pi)^{-m/2}\tfrac{1}{360}(60E_{;kk} + 60\tau E + 180E^2 + 12\tau_{;kk}\mathrm{Id}\\
&\quad +5\tau^2\mathrm{Id}\, - 2|\rho|^2\mathrm{Id}\, + 2|R|^2\mathrm{Id}\, + 30\Omega_{ij}\Omega_{ij})\,.
\end{aligned}$$

Lemma 1.3.6, and hence Lemma 1.7.7, follows from a detailed analysis of the Seeley calculus. We refer to Section 3.1.8 where we will give a second proof of Lemma 1.7.7 which uses only the fact that the invariants e_n and $e_{n,k}$ are locally computable, the fact that

$$e_n(x, c^2 D) = c^n e_n(x, D)$$

(see Theorem 3.1.10), and the relations of Display (1.7.h). It is convenient, however, to postpone this analysis for a bit and to have Lemma 1.7.7 available for use at the present moment for use in studying the supertrace asymptotics of the Witten Laplacian in the next section.

1.7.4 The restriction $r:\mathcal{P}_{n,m}\to\mathcal{P}_{n,m-1}$

Restricting the indices to range from 1 through $m-1$ instead of from 1 through m induces a natural *restriction map*

$$r:\mathcal{P}_{n,m}\to\mathcal{P}_{n,m-1}\,.$$

This map is surjective by Theorem 1.7.4. We can also describe the restriction map geometrically. If (N,g_N) is an $m-1$ dimensional Riemannian manifold, then we let $M:=N\times S^1$ and $g_M:=g_N+d\theta^2$. If $x\in N$ is the point of evaluation, we take the corresponding point $(x,1)\in M$ for evaluation; which point on the circle chosen is, of course, irrelevant as S^1 has a rotational symmetry. If $P\in\mathcal{P}_{n,m}$, then we have that

$$r(P)(g_N)(x)=P(g_N+d\theta^2)(x,1)\,.$$

Let $I=(i_1,...,i_k)$ and $J=(j_1,...,j_k)$ be collections of k indices. We define the *totally anti-symmetric tensor* ε by setting

$$\varepsilon^I_J:=g(e_{i_1}\wedge...\wedge e_{i_k},e_{j_1}\wedge...\wedge e_{j_k})\,. \tag{1.7.i}$$

Clearly ε vanishes if I and J are not permutations of the same set of k distinct indices; in particular $\varepsilon=0$ if $k>m$. If I and J are permutations of a set of k distinct indices, then ε is the sign of the permutation which sends I to J. By Theorem 1.7.4, $\mathcal{P}_{n,m}\cap\ker(r)$ is generated by invariants where we contract $2m$ indices using the ε tensor and contract the remaining indices in pairs. This observation will play a crucial role in the next section.

1.8 Applications of the second main theorem of invariance theory

In this section, we shall begin our study of the heat supertrace asymptotics of the Witten Laplacian by using the second main theorem of invariance theory to obtain spanning sets for the spaces of invariants which arise in this context. We will conclude our study of these invariants in Section 3.8, but it is convenient to present the formal properties of these invariants that depend on the second main theorem of invariance theory at this time.

We follow the discussion in [202, 203, 204]. Let $\phi\in C^\infty(M)$ be the dilaton. The *Witten Laplacian*, introduced in Section 1.2.6, is given by

$$\begin{aligned}&\Delta_\phi:=d_\phi\delta_\phi+\delta_\phi d_\phi\quad\text{where}\\&d_\phi:=e^{-\phi}de^\phi\quad\text{and}\quad\delta_\phi:=e^\phi\delta e^{-\phi}\,.\end{aligned}$$

1.8.1 The invariants $a^{d+\delta}_{n,m}$ for closed Riemannian manifolds

For the moment, let M be a compact smooth Riemannian manifold without boundary. Let $a_n(x,\cdot)$ be the local heat trace asymptotic coefficients defined in Equation (1.3.c). If $f\in C^\infty(M)$ is an auxiliary scalar localizing or smearing function, then

$$\mathrm{Tr}_{L^2}(fe^{-t\Delta_\phi^p}) \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M f(x) a_n(x, \Delta_\phi^p) dx \,.$$

The *heat supertrace asymptotics* of the Witten Laplacian are defined by

$$a_{n,m}^{d+\delta}(\phi, g)(x) := \sum_{p=0}^{m} (-1)^p a_n(x, \Delta_\phi^p) \,.$$

The metric g and the dimension m are introduced into the notation for later convenience. Let f be a smooth scalar function on M. Then

$$\sum_p (-1)^p \mathrm{Tr}_{L^2}(fe^{-t\Delta_\phi^p}) \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M f(x) \cdot a_{n,m}^{d+\delta}(\phi, g)(x) dx \,.$$

We first establish a useful duality result:

Lemma 1.8.1 *1. We have* $a_n(x, \Delta_\phi^p) = a_n(x, \Delta_{-\phi}^{m-p})$.

2. We have $a_{n,m}^{d+\delta}(\phi, g) = (-1)^m a_{n,m}^{d+\delta}(-\phi, g)$.

3. We have $a_{n,m}^{d+\delta} = 0$ *if* n *is odd.*

Proof: Since the invariants in question are local, global questions of orientability play no role and we may without loss of generality assume that the manifold in question is orientable. Let $\tilde\star$ be the normalized Hodge star operator defined in Equation (1.2.f). We have $\tilde\star^2 = \mathrm{Id}$ and, by Lemma 1.2.8, $\tilde\star \Delta_\phi \tilde\star = \Delta_{-\phi}$. Once the grading is taken into account, we have

$$\tilde\star \Delta_\phi^p \tilde\star = \Delta_{-\phi}^{m-p} \,.$$

Assertions (1) and (2) now follow. We may now use Theorem 1.3.5 to establish Assertion (3). $\square$

There are some additional properties of these invariants that will be useful.

Lemma 1.8.2 *1.* $a_{0,m}^{d+\delta} = 0$.

2. Let $M := M_1 \times M_2$ *where* M_i *are closed Riemannian manifolds of dimensions* m_i. *Give* M *a product metric* $g_M = g_1 + g_2$. *Let* $\phi_M = \phi_1 + \phi_2$ *where* ϕ_i *are smooth functions on* M_i. *Then:*

(a) $a_n(\Delta_\phi^p) = \sum_{n=n_1+n_2, p=p_1+p_2} a_{n_1}(\Delta_{\phi_1}^{p_1}) a_{n_2}(\Delta_{\phi_2}^{p_2})$.

(b) $a_{n,m}^{d+\delta}(\phi, g) = \sum_{n=n_1+n_2} a_{n_1,m_1}^{d+\delta}(\phi_1, g_1) a_{n_2,m_2}^{d+\delta}(\phi_2, g_2)$.

Proof: Since $a_{0,m}^{d+\delta}$ is homogeneous of weight 0 in the derivatives of ϕ and g, there is a universal constant so that $a_{0,m}^{d+\delta}(\phi, g) = c_m$. We use Lemma 1.3.10 to prove Assertion (1) by showing

$$0 = \int_M a_{0,m}^{d+\delta}(\phi, g) dx = \int_M c_m dx \quad \text{so} \quad c_m = 0 \,.$$

If $M = M_1 \times M_2$, then we may decompose

$$\Lambda^p(M) = \oplus_{p=p_1+p_2}\Lambda^{p_1}(M_1) \otimes \Lambda^{p_2}(M_2)\,.$$

Introduce the parity operator

$$\varepsilon := (-1)^{p_1}\mathrm{Id} \quad \text{on} \quad \Lambda^{p_1}(M_1)\,.$$

Since M is given the product structures, we may decompose

$$d_\phi = d_{\phi_1} \otimes \mathrm{Id} + \varepsilon \otimes d_{\phi_2} \quad \text{and} \quad \delta_\phi = \delta_{\phi_1} \otimes \mathrm{Id} + \varepsilon \otimes \delta_{\phi_2}\,.$$

The presence of ε ensures the operators

$$\{d_{\phi_1} \otimes \mathrm{Id}\,,\ \delta_{\phi_1} \otimes \mathrm{Id}\,,\ \varepsilon \otimes d_{\phi_2},\ \varepsilon \otimes \delta_{\phi_2}\}$$

anti-commute. Thus we may express

$$\Delta_\phi^p = \oplus_{p=p_1+p_2}\left\{\Delta_{\phi_1}^{p_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{\phi_2}^{p_2}\right\}\,.$$

This induces a corresponding splitting of the fundamental solution of the heat equation

$$e^{-t\Delta_\phi^p} = \oplus_{p=p_1+p_2} e^{-t\Delta_{\phi_1}^{p_1}} \otimes e^{-t\Delta_{\phi_2}^{p_2}}\,.$$

Let $f = f_1 f_2$ where $f_i \in C^\infty(M_i)$. We then have

$$\mathrm{Tr}_{L^2}\left\{fe^{-t\Delta_\phi^p}\right\} = \sum_{p=p_1+p_2}\mathrm{Tr}_{L^2}\left\{f_1 e^{-t\Delta_{\phi_1}^{p_1}}\right\} \cdot \mathrm{Tr}_{L^2}\left\{f_2 e^{-t\Delta_{\phi_2}^{p_2}}\right\}\,.$$

This implies a corresponding splitting of the local heat trace integrands

$$a_n(x, \Delta_\phi^p) = \sum_{n=n_1+n_2}\ \sum_{p=p_1+p_2} a_{n_1}(x_1, \Delta_{\phi_1}^{p_1}) \cdot a_{n_2}(x_2, \Delta_{\phi_2}^{p_2})$$

which proves the first identity of Assertion (2). We use this identity to complete the proof of Assertion (2) by computing

$$\begin{aligned}
a_{n,m}^{d+\delta}(\phi, g)(x) &= \sum_{p=0}^{m}(-1)^p a_n(x, \Delta_\phi^p) \\
&= \sum_{n_1+n_2=n}\ \sum_{p=0}^{m}(-1)^p \sum_{p=p_1+p_2} a_{n_1}(x_1, \Delta_{\phi_1}^{p_1}) \cdot a_{n_2}(x_2, \Delta_{\phi_2}^{p_2}) \\
&= \sum_{n=n_1+n_2}\left\{\sum_{p_1=0}^{m_1}(-1)^{p_1} a_{n_1}(x_1, \Delta_{\phi_1}^{p_1})\right\} \cdot \left\{\sum_{p_2=0}^{m_2}(-1)^{p_2} a_{n_2}(x_2, \Delta_{\phi_2}^{p_2})\right\} \\
&= \sum_{n=n_1+n_2} a_{n_1,m_1}^{d+\delta}(\phi_1, g_1)(x_1) \cdot a_{n_2,m_2}^{d+\delta}(\phi_2, g_2)\,. \qquad \square
\end{aligned}$$

1.8.2 Invariance theory

Let $\mathcal{Q}_m$ be the space of all $O(m)$ invariant polynomials in the components of the tensors

$$\{R,\ \nabla R,\ \nabla^2 R, \ldots,\ \nabla\phi,\ \nabla^2\phi, \ldots\}\,.$$

We permit only monomials which either do not involve ϕ or which involve at least two covariant derivatives of ϕ. We define

$$\text{weight}\,(\nabla^k\phi) = k \quad \text{and} \quad \text{weight}\,(\nabla^k R) = 2 + k$$

and let $\mathcal{Q}_{n,m}$ be the subspace of invariant polynomials of total weight n;

$$\mathcal{Q}_{n,m} = \{Q \in \mathcal{Q}_m : Q(\phi, c^2 g) = c^{-n} Q(\phi, g)\ \forall 0 \neq c \in \mathbb{R}\}\,.$$

We use the $\mathbb{Z}_2$ action $\phi \to -\phi$ to decompose

$$\begin{aligned}
\mathcal{Q}_{n,m} &= \mathcal{Q}^+_{n,m} \oplus \mathcal{Q}^-_{n,m} \quad \text{where} \\
\mathcal{Q}^\pm_{n,m} &:= \{Q \in \mathcal{Q}_{n,m} : Q(\phi, g) = \pm Q(-\phi, g)\}\,.
\end{aligned}$$

The restriction map r of Section 1.7.4 extends naturally to surjective maps

$$r : \mathcal{Q}^\pm_{n,m} \to \mathcal{Q}^\pm_{n,m-1} \to 0\,.$$

Lemma 1.8.3 *1. If m is even, then $a^{d+\delta}_{n,m}(\phi, g) \in \mathcal{Q}^+_{n,m} \cap \ker r$.*

2. If m is odd, then $a^{d+\delta}_{n,m}(\phi, g) \in \mathcal{Q}^-_{n,m} \cap \ker r$ and $a^{d+\delta}_{n,m}(0, g) = 0$.

Proof: We use Lemma 1.7.7 to see that the invariants $a^{d+\delta}_{n,m}(\phi, g)$ are homogeneous of weight n in the jets of the metric and of ϕ. Let ∇ be the Levi-Civita connection on $\Lambda(M)$. By Lemma 1.2.8,

$$\begin{aligned}
\Delta^p_\phi &= -\mathrm{Tr}\,(\nabla^2 + E^p_\phi) \quad \text{for} \\
E_\phi &= -\tfrac{1}{2}\gamma_i\gamma_j\Omega_{ij} - \phi_{;i}\phi_{;i} - \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i)\,.
\end{aligned}$$

Thus the undifferentiated variable ϕ does not play a role in these invariants. Furthermore, either at least 2 covariant derivatives of ϕ appear or only the curvature R appears in each *Weyl monomial* of $a^{d+\delta}_{n,m}(\phi, g)$. This shows that

$$a^{d+\delta}_{n,m}(\phi, g) \in \mathcal{Q}_{n,m}\,.$$

We use Lemma 1.8.1 to see that $a^{d+\delta}_{n,m}(\phi, g)$ is an odd function of ϕ if m is odd and an even function of ϕ if m is even. In particular, $a^{d+\delta}_{n,m}(0, g)$ must vanish if m is odd.

To complete the proof, we must show $ra^{d+\delta}_{n,m} = 0$. Give $M = N \times S^1$ the product metric and let $\phi = \phi_N$ be independent of the angular parameter $\theta \in S^1$. As the metric is flat on the circle and as $\phi_{S^1} = 0$,

$$a^{d+\delta}_{n,1}(\phi_{S^1}, g_{S^1}) = 0 \quad \text{for} \quad n > 0\,.$$

By Lemma 1.8.2 (1),

$$a^{d+\delta}_{0,1}(\phi_{S^1}, g_{S^1}) = 0\,.$$

Thus Lemma 1.8.2 (2) implies $a^{d+\delta}_{n,m}(\phi_M, g_M) = 0$ so $ra^{d+\delta}_{n,m} = 0$. $\square$

Let ε be the totally anti-symmetric tensor defined in Equation (1.7.i). Let I and J be m tuples of indices indexing an orthonormal frame for $T(M)$. Let

$$\mathcal{R}^{I,t}_{J,s} := R_{i_s i_{s+1} j_{s+1} j_s} \ldots R_{i_{t-1} i_t j_t j_{t-1}}\,.$$

By definition, the empty product is 1. Consequently if $t < s$ or if $t - s$ is even, then we shall set $\mathcal{R}_{B,s}^{A,t} = 1$. The following Lemma will be used to study $a_{n,m}^{d+\delta}$ for $n \leq m + 1$.

Lemma 1.8.4 *1. If $n < m$, then $\mathcal{Q}_{n,m} \cap \ker r = \{0\}$.*

2. If m is even, then $\mathcal{Q}_{m,m} \cap \ker r = \text{Span}\,\{\varepsilon_J^I \mathcal{R}_{J,1}^{I,m}\}$.

3. If m is odd, then $\mathcal{Q}_{m+1,m}^{-} \cap \ker r = \text{Span}\,\{\varepsilon_J^I \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{I,m}\}$.

Proof: We follow the arguments given in [202, 204] which generalized previous results of [173] dealing with the case $\phi = 0$. Let $0 \neq Q \in \mathcal{Q}_{n,m} \cap \ker r$. Let A be a monomial of Q of the form

$$A = \phi_{;\alpha_1} \dots \phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1;\beta_1} \dots R_{i_v j_v k_v \ell_v;\beta_v}$$

where α_μ and β_ν denote appropriate collections of indices. The weight n of A is then given by

$$n = \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (2 + |\beta_\nu|)\,.$$

By definition, the empty sum is 0. Thus $\sum_\mu$ is to be ignored if $u = 0$. Similarly $\sum_\nu$ is to be ignored if $v = 0$. Let k be total number of indices present in A. We then have

$$k = \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (4 + |\beta_\nu|) = n + 2v\,.$$

The same argument extending Theorem 1.7.3 from the algebraic to the geometric context can be used to extend Theorem 1.7.4 from the algebraic to the geometric context. To ensure that $rQ = 0$, we must contract $2m$ indices in A using the ε tensor and then contract the remaining indices of A in pairs. Since at least $2m$ indices must appear in A,

$$2m \leq k = n + 2v = 2n - \sum_{\mu=1}^{u} |\alpha_\mu| - \sum_{\nu=1}^{v} |\beta_\nu| \leq 2n\,. \qquad (1.8.a)$$

This is not possible, of course, if $n < m$ and Assertion (1) now follows.

If $n = m$ is even, then all of the inequalities in Display (1.8.a) must have been equalities. Thus there are no covariant derivatives and the ϕ variables do not appear. Furthermore, all indices are contracted using the ε tensor. The first Bianchi identity given in Display (1.1.b) shows we need not alternate 3 indices in a given R variable. Thus we need consider only expressions $R_{i_1 i_2 j_2 j_1}$ or $R_{i_1 j_1 i_2 j_2}$ in A. By the first Bianchi identity,

$$R_{i_1 j_1 i_2 j_2} - R_{i_1 j_2 i_2 j_1} = -R_{i_1 i_2 j_2 j_1}\,.$$

This implies that

$$\varepsilon_J^I \dots R_{i_1 j_1 i_2 j_2} \dots = -\tfrac{1}{2} \varepsilon_J^I \dots R_{i_1 i_2 j_2 j_1} \dots\,.$$

Consequently $\mathcal{Q}_{m,m} \cap \ker r$ is spanned by the invariant $\varepsilon_J^I \mathcal{R}_{J,1}^{I,m}$. This proves Assertion (2).

In the proof of Assertion (3), we have

$$m = 2\bar{m} + 1 \quad \text{and} \quad n = m + 1\,.$$

Thus the inequalities of Display (1.8.a), $2m \le ... \le 2n = 2m + 2$, represent a total increase by 2. Since $2m$, $n + 2v$, and $2n$ are all even, only one of the two inequalities given in Display (1.8.a) can be strict. As $Q(-\phi, g) = -Q(\phi, g)$, u must be *odd.* Thus

$$\sum_{\mu=1}^{u} |\alpha_\mu| > 0$$

so the second inequality in Equation (1.8.a) is strict. Consequently

$$k = 2m \quad \text{and} \quad \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} |\beta_\nu| = 2\,.$$

Since $u > 0$, $\sum_\mu |\alpha_\mu| > 0$. Since ϕ appears, at least 2 covariant derivatives of ϕ appear. Thus $|\beta_\nu| = 0$ for all ν, $u = 1$, and $|\alpha_1| = 2$. Since $k = 2m$, each index is contracted using the ε tensor. The argument given above shows we can write $R_{i_1j_1i_2j_2}$ in terms of $R_{i_1i_2j_2j_1}$. Thus we are in fact dealing with a multiple of $\varepsilon^I_J\phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}$, which proves the final Assertion. $\square$

A similar argument could be used to show, after a fair amount of additional work to eliminate dependencies, that if m is even, then

$$\begin{aligned}
\mathcal{Q}^+_{m+2,m} \cap \ker r = \operatorname{Span}\{&\varepsilon^I_J\phi_{;i_1j_1}\phi_{;i_2j_2}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_J\phi_{;k}\phi_{;k}\mathcal{R}^{I,m}_{J,1},\\
&\varepsilon^I_JR_{k\ell\ell k}\mathcal{R}^{I,m}_{J,1},\ \varepsilon^I_J\phi_{;i_1}\phi_{;j_1}R_{ki_2j_2k}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{i_1i_2j_2j_1;kk}\mathcal{R}^{I,m}_{J,3},\\
&\varepsilon^I_JR_{i_1i_2j_2j_1;k}R_{i_3i_4j_4j_3;k}\mathcal{R}^{I,m}_{J,5}, \varepsilon^I_JR_{ki_1j_1k}R_{\ell i_2j_2\ell}\mathcal{R}^{I,m}_{J,3},\\
&\varepsilon^I_JR_{k\ell j_2j_1}R_{i_1i_2k\ell}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{ki_1j_1\ell}R_{ki_2j_2\ell}\mathcal{R}^{I,m}_{J,3},\\
&\varepsilon^I_JR_{ki_1j_2j_1}R_{kj_3i_3i_2}R_{\ell i_4j_5j_4}R_{\ell j_6i_6i_5}\mathcal{R}^{I,m}_{J,7}\}.
\end{aligned} \tag{1.8.b}$$

We omit certain invariants from the list if m is small.

1.8.3 Formal cohomology groups of spaces of invariants

Although Equation (1.8.b) would simplify somewhat the study of $a^{d+\delta}_{m+2,m}$, the list of invariants is still rather long. Thus we use a different approach to the matter using the observation from Lemma 1.3.10 that $\int_M a^{d+\delta}_{m+2,m}(\phi, g)dx = 0$.

Let $\mathcal{Q}^{p,+}_{n,m}$ be the space of p form valued invariants in the curvature tensor, the covariant derivatives of the curvature tensor, and the covariant derivatives of ϕ which are even in ϕ. The ordinary exterior co-derivative δ induces a natural map

$$\delta : \mathcal{Q}^{p,+}_{n,m} \to \mathcal{Q}^{p-1,+}_{n+1,m}\,.$$

(We do not use the Witten co-derivative at this stage.) In contrast to the case $p = 0$, we do not impose the requirement that ϕ does not appear. We also do not impose the requirement that either no covariant derivatives of ϕ appear or at least 2 covariant derivatives of ϕ appear. This is a crucial technical point.

Let N be a Riemannian manifold of dimension $m-1$. Let $M := N \times S^1$ and let

$$i : N \to N \times \{1\} \subset M$$

be the natural inclusion. The restriction map

$$r : \mathcal{Q}^{p,+}_{n,m} \to \mathcal{Q}^{p,+}_{n,m-1}$$

is defined dually by the identity:

$$rQ(\phi_N, g_N) = i^* Q(\phi_N, g_N + d\theta^2)\,.$$

Lemma 1.8.5

1. *The map $r : \mathcal{Q}^{p,+}_{n,m} \to \mathcal{Q}^{p,+}_{n,m-1}$ is surjective.*
2. *We have $r\delta^M = \delta^N r$.*

Proof: It is straightforward to extend Theorem 1.7.3 from the context of scalar invariants to p form valued invariants. We refer to [7] for further details. Including the dilaton ϕ in the discussion also poses no difficulties. Thus one may show that all p form valued invariants are constructed by alternating p indices and by contracting the remaining indices in pairs in monomials formed from the covariant derivatives of R and of ϕ. The invariants that belong to $\mathcal{Q}^p_{n,m}$ are defined by letting the indices in question range from 1 to m. We define rQ by restricting the range of summation to be from 1 to $m-1$. Assertion (1) now follows.

Let $\{e_1, ..., e_m\}$ be a local orthonormal frame for the tangent bundle of $M := N \times S^1$. Assume $e_m = \partial_\theta$. As the structures are flat in the S^1 direction,

$$\begin{aligned}
&\nabla^M_{e_m} q(g_N + d\theta^2, \phi_N) = 0,\\
&i^*\nabla^M_{e_a} q(g_N + d\theta^2, \phi_N) = \nabla^N_{e_a} i^* q(g_N + d\theta^2, \phi_N),\\
&i^*\mathfrak{i}(e_a) = \mathfrak{i}(e_a) i^*\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\delta^N r(q)(g_N, \phi_N) = -\mathfrak{i}(e_a)\nabla^N_{e_a} i^* q(g_N + d\theta^2, \phi_N)\\
= \quad & -i^*\{\mathfrak{i}(e_a)\nabla^M_{e_a} + \mathfrak{i}(e_m)\nabla^M_{e_m}\} q(g_N + d\theta^2, \phi_N)\\
= \quad & i^*\delta^M q(g_N + d\theta^2, \phi_N) = r(\delta^M q)(g_N, \phi_N)\,.
\end{aligned}$$

This establishes Assertion (2). $\square$

If $P \in \mathcal{P}_{n,m}$, then the evaluation $\mathfrak{I}(P)(g) \in \mathbb{R}$ is defined by setting

$$\mathfrak{I}P(\phi, g) := \int_{\partial M} P(\phi, g)(y)dy\,.$$

The analysis of the formal cohomology groups of the spaces of invariants of Riemannian manifolds, which was given in [172], then extends immediately to this more general setting to yield:

Lemma 1.8.6

1. *If* $Q \in \mathcal{Q}^{+}_{n,m} \cap \ker \mathfrak{I}$ *and if* $n \neq m$, *then there exists* $Q^1 \in \mathcal{Q}^{1,+}_{n-1,m}$ *so that* $\delta Q^1 = Q$.
2. *If* $Q^1 \in \mathcal{Q}^{1,+}_{n,m}$, *if* $\delta Q^1 = 0$, *and if* $n \neq m-1$, *then there exists* $Q^2 \in \mathcal{Q}^{2,+}_{n-1,m}$ *so that* $Q^1 = \delta Q^2$.

Assertion (1) shows that any scalar invariant which always integrates to zero is canonically a divergence. Assertion (2) implies that any 1 form valued invariant which is co-closed is canonically co-exact. The restriction on the weight n is a technical one which causes no difficulty as we shall be taking $n = m+2$ in Assertion (1) and we shall be taking $n = m+1$ in Assertion (2).

We use this result to show

Lemma 1.8.7 *If* m *is even, then there exists a* 1 *form valued invariant* $Q^1_{m+1,m}$ *in* $\mathcal{Q}^{1,+}_{m+1,m} \cap \ker r$ *so that* $\delta Q^1_{m+1,m} = a^{d+\delta}_{m+2,m}$.

Proof: By Lemma 1.3.10, $\mathfrak{I}(a^{d+\delta}_{m+2,m}(\phi,g)) = 0$. Thus Lemma 1.8.6 (1), implies that there exists

$$\bar{Q}^1_{m+1,m} \in \mathcal{Q}^{1,+}_{m+1,m} \quad \text{so} \quad \delta\bar{Q}^1_{m+1,m}(\phi,g) = a^{d+\delta}_{m+2,m}(\phi,g)\,.$$

Unfortunately, $r\bar{Q}^1_{m+1,m}$ need not be zero and we must correct for this. Since $ra^{d+\delta}_{m+2,m} = 0$, we use Lemma 1.8.5 (2) to see

$$0 = r\delta\bar{Q}^1_{m+1,m} = \delta r\bar{Q}^1_{m+1,m}\,.$$

Thus by Lemma 1.8.6 (2), there exists

$$\bar{Q}^2_{m,m-1} \in \mathcal{Q}^{2,+}_{m,m-1} \quad \text{so} \quad r\bar{Q}^1_{m+1,m} = \delta\bar{Q}^2_{m,m-1}\,.$$

Since by Lemma 1.8.5 r is surjective, there exists

$$Q^2_{m,m} \in \mathcal{Q}^{2,+}_{m,m} \quad \text{so} \quad rQ^2_{m,m} = \bar{Q}^2_{m,m-1}\,.$$

We complete the proof by setting $Q^1_{m+1,m} := \bar{Q}^1_{m+1,m} - \delta Q^2_{m,m}$. Then

$$\begin{aligned}
\delta Q^1_{m+1,m} &= \delta\bar{Q}^1_{m+1,m} - \delta\delta Q^2_{m,m} = \delta\bar{Q}^1_{m+1,m} = a^{d+\delta}_{m+2,m} \\
rQ^1_{m+1,m} &= r\bar{Q}^1_{m+1,m} - r\delta Q^2_{m,m} = r\bar{Q}^1_{m+1,m} - \delta rQ^2_{m,m} \\
&= r\bar{Q}^1_{m+1,m} - \delta\bar{Q}^2_{m,m-1} = 0\,.
\end{aligned}$$

Thus $Q^1_{m+1,m}$ has the required properties. $\square$

For m even, we define elements of $\mathcal{Q}^{1,+}_{m+1,m} \cap \ker r$ by setting:

$$\begin{aligned}
\Xi^{1,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3}e^{j_2} && (\ell \text{ even}),\\
\Xi^{2,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3}e^{k} && (\ell \text{ even}),\\
\Xi^{3,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3}e^{j_2} && (\ell \text{ even}),\\
\Xi^{4,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;k}\mathcal{R}^{I,m}_{J,1}e^{k}, && (\ell \text{ odd}),\\
\Xi^{5,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;i_1}R_{i_2kkj_2}\mathcal{R}^{I,m}_{J,3}e^{j_1} && (\ell \text{ odd}).
\end{aligned}$$

Lemma 1.8.8 *Let $m = 2\bar{m}$. Then $\mathcal{Q}^{1,+}_{m+1,m} \cap \ker r = \operatorname{Span}_{i,\ell}\{\Xi^{i,\ell}_{m+1,m}\}$.*

Proof: Let $0 \neq Q_{m+1,m} \in \mathcal{Q}^{1,+}_{m+1,m} \cap \ker r$. Let A be a monomial of $Q_{m+1,m}$. Expand

$$A = \phi^\ell \phi_{;\alpha_1} ... \phi_{;\alpha_u} R_{i_1 j_1 k_1 l_1;\beta_1} ... R_{i_v j_v k_v l_v;\beta_v} e^h$$

where $|\alpha_\nu| \geq 1$ and $\ell + u$ is even. The weight n of A is given by

$$n = \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (|\beta_\nu| + 2)\,.$$

To ensure that $rQ = 0$, we must use the ε tensor to contract $2m$ indices in A and contract the remaining indices in pairs using the inner product. Let k be the total number of indices appearing in A. As in the proof of Lemma 1.8.4, the following estimate is critical:

$$\begin{aligned} 2m \quad &\leq \quad k = \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (|4 + \beta_\nu|) + 1 = n + 2v + 1 && (1.8.c) \\ &= \quad 2n + 1 - \sum_{\mu=1}^{u} |\alpha_\mu| - \sum_{\nu=1}^{v} |\beta_\nu| \leq 2n + 1 && (1.8.d) \end{aligned}$$

We set $n = m + 1$. These inequalities, $2m \leq ... \leq 2n + 1 = 2m + 3$, represent a total increase by 3. Since $2m$ and $n + 2v + 1$ are both even, the inequality in Display (1.8.d) must be strict and represent an increase either of 1 or of 3. We distinguish two cases:

Case 1: Suppose that Display (1.8.c) is an equality. Then all the $2m$ indices present in A are contracted using the ε tensor. We can commute covariant derivatives at the cost of introducing additional curvature terms. Thus since all indices are to be contracted using the ε tensor, we do not alternate two indices in $\phi_{;...}$. This means that we may assume without loss of generality that the variables

$$\phi_{;...i_1...i_2...} \quad \text{and} \quad \phi_{;...j_1...j_2...}$$

are not present. This implies that $|\alpha_\mu| \leq 2$ for all μ. Furthermore, by the first and second Bianchi identity, at most 2 indices can be alternated in $R_{ijkl;\beta}$. Thus $|\beta_\nu| = 0$ for all ν. Since the inequality in Display (1.8.d) represents an increase of 3, we have $\sum_{\mu=1}^{u} |\alpha_\mu| = 3$. This leads to the invariants which have the form $\Xi^{1,\ell}_{m+1,m}$.

Case 2 Suppose that Display (1.8.c) is not an equality. Then there are $2m + 2$ indices and one explicit covariant derivative present in A; $2m$ indices are contracted using the ε tensor and two indices are contracted as a pair. This yields the invariants $\Xi^{i,\ell}_{m+1,m}$ for $i = 2, 3, 4, 5$ and the additional invariants:

$$\begin{aligned} \Theta^{1,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;k} R_{i_1 i_2 j_2 k} \mathcal{R}^{I,m}_{J,3} e^{j_1} && (\ell \text{ odd}), \\ \Theta^{2,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;i_1} R_{i_2 k j_2 j_1} \mathcal{R}^{I,m}_{J,3} e^{k} && (\ell \text{ odd}), \\ \Theta^{3,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell \phi_{;i_1} R_{i_2 k j_3 j_2} R_{i_3 i_4 j_4 k} \mathcal{R}^{I,m}_{J,5} e^{j_1} && (\ell \text{ odd}), \\ \Theta^{4,\ell}_{m+1,m} &:= \varepsilon^I_J \phi^\ell R_{i_1 i_2 k j_2} R_{i_3 i_4 j_4 j_3;k} \mathcal{R}^{I,m}_{J,5} e^{j_1} && (\ell \text{ even}). \end{aligned}$$

Note that the invariants $\Theta^{3,\ell}_{m+1,m}$ (resp. $\Theta^{4,\ell}_{m+1,m}$) are to be set to 0 if $m<3$ (resp. $m<4$).

To complete the proof, we must show the invariants $\Theta^{i,\ell}_{m+1,m}$ play no role. At this stage, we use the vanishing

$$0=\det\begin{pmatrix} g(v_1,w_1) & g(v_2,w_1) & \ldots & g(v_{m+1},w_1)\\ g(v_1,w_2) & g(v_2,w_2) & \ldots & g(v_{m+1},w_2)\\ \ldots & \ldots & \ldots & \ldots\\ g(v_1,w_{m+1}) & g(v_2,w_{m+1}) & \ldots & g(v_{m+1},w_{m+1})\end{pmatrix},$$

which played a central role in Theorem 1.7.2, to derive some "unexpected relationships". Similar arguments were employed to establish Equation (1.8.b) and will be used subsequently in the proof of Lemma 1.8.10.

Let $U=(u_1,\ldots,u_{m+1})$ and let $V=(v_1,\ldots,v_{m+1})$ be collections of $m+1$ indices. Since $\varepsilon^U_V=0$, we have

$$0=\varepsilon^U_V\phi^\ell\phi_{;u_1}\mathcal{R}^{U,m+1}_{V,2}e^{v_1}, \tag{1.8.e}$$

where we sum over all possible pairs of $m+1$ tuples U and V.

We shall expand the determinant ε^U_V in minors around various rows and columns. Let $u_1=k$ and let $I=(i_1,\ldots,i_m):=(u_2,\ldots,u_{m+1})$. We expand around row 1 and column 1 setting

$$v_1=k\quad\text{and}\quad J=(j_1,\ldots,j_m):=(v_2,\ldots,v_{m+1})\,.$$

This yields the invariant

$$\varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^k\,.$$

Then we set $v_2=k$ and $J=(j_1,\ldots,j_m):=(v_1,v_3,\ldots,v_{m+1})$ to obtain, after taking into account the fact that the relevant sign is -1 when expanding a determinant in minors around row 1 and column 2, the invariant

$$-\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\,.$$

Then we set $v_3=k$ and $J=(j_1,\ldots,j_m):=(v_1,v_2,v_4,\ldots,v_{m+1})$. Since we are expanding in minors using the row 1 and column 3, the relevant sign is $+$. This creates the invariant

$$+\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2kj_2}\mathcal{R}^{I,m}_{J,3}e^{j_1}=-\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\,.$$

We continue in this fashion until finally we have that $v_{m+1}=k$ and we have that $J=(j_1,\ldots,j_m):=(v_1,\ldots,v_m)$ which creates the invariant

$$+\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_3j_2}\ldots R_{i_{m-1}i_mkj_m}e^{j_1}=-\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\,.$$

We sum these invariants to expand Equation (1.8.e) around row 1 to show

$$\begin{aligned} 0 &= \varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^k-m\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1} \\ &= \Xi^{4,\ell}_{m+1,m}-m\Theta^{1,\ell}_{m+1,m}\,. \end{aligned} \tag{1.8.f}$$

Next, we expand about row 2. We set $u_2=k$ and argue as above to see

$$0 = \varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2j_1}\mathcal{R}^{I,m}_{J,3}e^k-2\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1} \tag{1.8.g}$$

$$
\begin{aligned}
& - \quad (m-2)\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_3j_2}R_{i_3i_4j_4k}\mathcal{R}^{I,m}_{J,5}e^{j_1}\\
& = \quad \Theta^{2,\ell}_{m+1,m}+2\Xi^{5,\ell}_{m+1,m}-(m-2)\Theta^{3,\ell}_{m+1,m}\,.
\end{aligned}
$$

Now we change the procedure slightly. We expand about column 1 by setting $v_1 = k$. A similar expansion by taking $u_1 = k$ for row 1, $u_2 = k$ for row 2, and so forth then yields the identity

$$
\begin{aligned}
0 &= \varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^k + m\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2j_1}\mathcal{R}^{I,m}_{J,3}e^k \\
&= \Xi^{4,\ell}_{m+1,m} + m\Theta^{2,\ell}_{m+1,m}\,.
\end{aligned}
\tag{1.8.h}
$$

We use Equations (1.8.f), (1.8.g), and (1.8.h) to conclude therefore

$$\{\Theta^{1,\ell}_{m+1,m},\Theta^{2,\ell}_{m+1,m},\Theta^{3,\ell}_{m+1,m}\}\subset \operatorname{Span}_{i,\ell}\{\Xi^{i,\ell}_{m+1,m}\}\,.$$

Finally, consider the identity

$$0=\varepsilon^U_V\phi^\ell R_{u_2u_3v_3v_2;u_1}\mathcal{R}^{U,m+1}_{V,4}e^{v_1}\,.$$

We expand around the first row. We set $u_1 = k$ and then set $v_1 = k$, $v_2 = k$, ..., $v_{m+1} = k$ in turn to derive the identity

$$
\begin{aligned}
0 &= \varepsilon^I_J\phi^\ell R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3}e^k - 2\varepsilon^I_J\phi^\ell R_{i_1i_2j_2k;k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\\
&\quad -(m-2)\varepsilon^I_J\phi^\ell R_{i_1i_2j_3j_2;k}R_{i_3i_4j_4k}\mathcal{R}^{I,m}_{J,5}e^{j_1}\\
&= \Xi^{2,\ell}_{m+1,m}-2\Xi^{3,\ell}_{m+1,m}+(m-2)\Theta^{4,\ell}_{m+1,m}\,.
\end{aligned}
$$

This establishes the lemma. □

1.8.4 Expressing the supertrace invariants in terms of a Weyl basis

We use the results of the previous Section to show

Lemma 1.8.9 *There exist universal constants so that*

1. *If $n < m$ or if n is odd, then $a^{d+\delta}_{n,m} = 0$.*
2. *If m is even, then $a^{d+\delta}_{m,m} = c_{m,m}\varepsilon^I_J\mathcal{R}^{I,m}_{J,1}$.*
3. *If m is odd, then $a^{d+\delta}_{m+1,m} = c_{m+1,m}\varepsilon^I_J\phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}$.*
4. *If m is even, then $a^{d+\delta}_{m+2,m} = c^1_{m+2,m}(\varepsilon^I_J\phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3})_{;j_2}$ $+c^2_{m+2,m}(\varepsilon^I_J\mathcal{R}^{I,m}_{J,1})_{;kk}+c^3_{m+2,m}(\varepsilon^I_J R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3})_{;j_2}$.*

Proof: Assertions (1), (2), and (3) follow directly from Lemmas 1.8.1, 1.8.2, 1.8.3, and 1.8.4. To prove Assertion (4), let m be even. We apply Lemma 1.8.7 and Lemma 1.8.8 to see there exist universal constants so

$$
\begin{aligned}
a^{d+\delta}_{m+2,m}(\phi,g) &= \sum_{i=1}^{3}\sum_{\ell-\text{even}} c^{i,\ell}_{m+1,m}\delta\Xi^{i,\ell}_{m+1,m}\\
&+ \sum_{i=4}^{5}\sum_{\ell-\text{odd}} c^{i,\ell}_{m+1,m}\delta\Xi^{i,\ell}_{m+1,m}\,.
\end{aligned}
\tag{1.8.i}
$$

We consider various contributions to $a^{d+\delta}_{m+2,m}$. We shall eliminate some monomials using the fact that the undifferentiated variable ϕ does not appear in any monomial of $a^{d+\delta}_{m+2,m}$. We will also use the fact that if a monomial of $a^{d+\delta}_{m+2,m}$ involves the jets of ϕ, then the weight of that monomial in ϕ is at least 2 to eliminate other monomials.

We first study terms which are linear in the 2 jets of ϕ and which have total weight 2 in ϕ. Such terms can arise only when $i = 4$ or when $i = 5$ in Equation (1.8.i). We indicate the remaining terms by $+\dots$ to express

$$a^{d+\delta}_{m+2,m}(\phi, g) = -\sum_{\ell-\text{odd}} \phi^\ell (c^{4,\ell}_{m+1,m} Q^4_{m+2,m} + c^{5,\ell}_{m+1,m} Q^5_{m+2,m}) + \dots \text{ for}$$

$$Q^4_{m+2,m} := \varepsilon^I_J \phi_{;kk} \mathcal{R}^{I,m}_{J,1} \quad \text{and} \quad Q^5_{m+2,m} := \varepsilon^I_J \phi_{;i_1 j_1} R_{i_2 kk j_2} \mathcal{R}^{I,m}_{J,3}\,.$$

Since $\ell > 0$, such terms do not appear in $a^{d+\delta}_{m+2,m}$. Consequently

$$0 = c^{4,\ell}_{m+1,m} Q^4_{m+2,m} + c^{5,\ell}_{m+1,m} Q^5_{m+2,m} \quad \text{for } \ell \text{ odd}\,.$$

We show that $Q^4_{m+2,m}$ and $Q^5_{m+2,m}$ are linearly independent by expanding

$$\begin{aligned}
Q^4_{m+2,m} &= \star \cdot A^4_{m+1,m} + 0 \cdot A^5_{m+1,m} + \dots \\
Q^5_{m+2,m} &= \star \cdot A^4_{m+1,m} + \star \cdot A^5_{m+1,m} + \dots. \quad \text{where} \\
A^4_{m+1,m} &:= \phi_{;11} R_{1221} R_{3443} \dots R_{m-1,mm,m-1} \quad \text{and} \\
A^5_{m+1,m} &:= \phi_{;12} R_{1332} R_{3443} \dots R_{m-1,mm,m-1}
\end{aligned}$$

and where $\star$ denotes a suitably chosen non-zero coefficient. This shows $Q^4_{m+2,m}$ and $Q^5_{m+2,m}$ are linearly independent so $c^{4,\ell}_{m+1,m} = c^{5,\ell}_{m+1,m} = 0$. Consequently

$$a^{d+\delta}_{m+2,m} = \sum_{i=1}^{3} \sum_{\ell-\text{even}} c^{i,\ell}_{m+1,m} \delta \Xi^{i,\ell}_{m+1,m}\,.$$

We may now argue that

$$\begin{aligned}
a^{d+\delta}_{n,m} &= -\sum_{\ell-\text{even}\,,\ell>0} \ell \phi^{\ell-1} \sum_{i=1}^{3} c^{i,\ell}_{m+1} Q^i_{m+2,m} + \dots \quad \text{where} \\
Q^1_{m+1,m} &:= \varepsilon^I_J \phi_{;j_2} \phi_{;i_1 j_1} \phi_{;i_2} \mathcal{R}^{I,m}_{J,3}, \\
Q^2_{m+1,m} &:= \varepsilon^I_J \phi_{;k} R_{i_1 i_2 j_2 j_1;k} \mathcal{R}^{I,m}_{J,3}, \\
Q^3_{m+1,m} &:= \varepsilon^I_J \phi_{;j_2} R_{i_1 i_2 k j_1;k} \mathcal{R}^{I,m}_{J,3}\,.
\end{aligned}$$

We show that these 3 invariants are linearly independent by computing

$$\begin{aligned}
Q^1_{m+1,m} &= \star A^1_{m+1,m} + 0 A^2_{m+1,m} + 0 A^3_{m+1,m} + \dots \\
Q^2_{m+1,m} &= 0 A^1_{m+1,m} + \star A^2_{m+1,m} + \star A^3_{m+1,m} + \dots \\
Q^3_{m+1,m} &= 0 A^1_{m+1,m} + \star A^2_{m+1,m} + 0 A^3_{m+1,m} + \dots \quad \text{where} \\
A^1_{m+1,m} &:= \phi_{;11} \phi^2_{;2} R_{3443} \dots R_{m-1,mm,m-1},
\end{aligned}$$

$$A^2_{m+1,m} := \phi_{;1} R_{1221;1} R_{3443} \ldots R_{m-1,mm,m-1}, \quad \text{and}$$
$$A^3_{m+1,m} := \phi_{;3} R_{1221;3} R_{3443} \ldots R_{m-1,mm,m-1} .$$

This shows that $c^{1,\ell}_{m+1,m} = c^{2,\ell}_{m+1,m} = c^{3,\ell}_{m+1,m} = 0$ for $\ell > 0$ and ℓ even. Lemma 1.8.9 now follows. □

Remark. It is by no means obvious that the invariants $\{Q^i_{m+2,m}\}_{i=1}^5$ form a linearly independent set. The proof of Lemma 1.8.8 shows that there are "unexpected relations". The use of the "classifying monomials" $A^i_{m+2,m}$ shows, fortunately, that this is not the case; there are no unexpected relations among the invariants $\{Q^i_{m+2,m}\}$.

1.8.5 Supertrace invariants for manifolds with boundary

Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M. Let $a_{n,k}$ be the heat trace invariants defined in Equation (1.4.e). Let $\mathcal{B} = \mathcal{B}_a$ or $\mathcal{B} = \mathcal{B}_r$ define either absolute or relative boundary conditions. Set

$$a^{d+\delta}_{n,m,k}(\phi, g; \mathcal{B})(y) := \sum_{p=0}^{m} (-1)^p a_{n,k}(y, \Delta^p_\phi, \mathcal{B}) .$$

If $f \in C^\infty(M)$, then

$$\begin{aligned} &\sum_{p=0}^{m} (-1)^p \operatorname{Tr}_{L^2}(f e^{-t\Delta^p_{\phi,\mathcal{B}}}) \\ \sim\ &\sum_{n=0}^{\infty} t^{(n-m)/2} \int_M f(x) \cdot a^{d+\delta}_{n,m}(\phi, g)(x) dx \\ +\ &\sum_{n=0}^{\infty} t^{(n-m)/2} \int_{\partial M} \sum_{k<n} \nabla^k_{e_m} f(y) \cdot a^{d+\delta}_{n,m,k}(\phi, g; \mathcal{B})(y) dy . \end{aligned}$$

The same argument given to establish Lemma 1.8.4 then shows

$$a^{d+\delta}_{n,m,k}(\phi, g; \mathcal{B}_a) = (-1)^m a^{d+\delta}_{n,m,k}(-\phi, g; \mathcal{B}_r) .$$

We therefore restrict to absolute boundary conditions and set

$$a^{d+\delta}_{n,m,k}(\phi, g) := a^{d+\delta}_{n,m,k}(\phi, g; \mathcal{B}_a) .$$

Let $\tilde\nabla$ be the Levi-Civita connection of the boundary. On the boundary, consider polynomials in the components of the tensors

$$\{R,\ \nabla R,\ \nabla^2 R,\ \ldots,\ L,\ \tilde\nabla L,\ \tilde\nabla^2 L,\ \ldots,\ \nabla\phi,\ \nabla^2\phi, \ldots\} .$$

We do not introduce the variable ϕ. We let

$$\begin{aligned} &\text{weight}\,(\nabla^k R) := 2 + k, \\ &\text{weight}\,(\tilde\nabla^k L) := 1 + k, \\ &\text{weight}\,(\nabla^k \phi) = k . \end{aligned}$$

Let $\tilde{\mathcal{Q}}_{n,m}$ be the space of all $O(m-1)$ invariant polynomials of total weight n where we admit monomials which either do not involve the covariant derivatives of ϕ at all or which involve at least two covariant derivatives of ϕ.

Lemma 1.8.3 extends immediately to this situation to show

$$a_{n,m,k}^{d+\delta} \in \tilde{\mathcal{Q}}_{n-k-1} \cap \ker r \, . \tag{1.8.j}$$

Let $\tilde{\mathcal{P}}_{n,m} \subset \tilde{\mathcal{Q}}_{n,m}$ be the subspace of invariants that do not involve the covariant derivatives of ϕ. Setting $\phi = 0$ defines a natural map from $\tilde{\mathcal{Q}}_{n,m}$ to $\tilde{\mathcal{P}}_{n,m}$. If $P \in \tilde{\mathcal{P}}_{n,m}$, then the evaluation $\mathfrak{I}(P)(g) \in \mathbb{R}$ is defined by setting

$$\mathfrak{I}P(g) := \int_{\partial M} P(g)(y)dy \, .$$

By Lemma 1.5.10,

$$0 = \int_M a_{m+1,m}^{d+\delta}(0,g)(x)dx + \int_{\partial M} a_{m+1,m,0}^{d+\delta}(0,g)(y)dy \, .$$

We apply Lemma 1.8.9. The interior integrand $a_{m+1,m}^{d+\delta}$ vanishes if m is even. It is divisible by $\nabla^2\phi$ if m is odd. Consequently it vanishes if $\phi = 0$, for either parity of m. This implies the boundary integral is zero and therefore

$$\mathfrak{I}a_{m+1,m,0}^{d+\delta}(g) = 0 \, . \tag{1.8.k}$$

Let A and B be $m-1$ tuples of indices indexing an orthonormal frame for $T(\partial M)$. Let ":" be covariant differentiation with respect to the Levi-Civita connection of ∂M. We define

$$\begin{aligned}
\mathcal{R}_{B,s}^{A,t} &:= R_{a_s a_{s+1} b_{s+1} b_s} \ldots R_{a_{t-1} a_t b_t b_{t-1}} \, , \\
\mathcal{L}_{B,s}^{A,t} &:= L_{a_s b_s} \ldots L_{a_t b_t} \, , \\
\mathcal{F}_{m-1,m}^{k} &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,m-1} \, , \\
\mathcal{F}_{m,m}^{1,k} &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \phi_{;a_{2k+1} b_{2k+1}} \mathcal{L}_{B,2k+2}^{A,m-1} \, , \\
\mathcal{F}_{m,m}^{2,k} &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \phi_{;a_{2k+1}} \phi_{;b_{2k+1}} \mathcal{L}_{B,2k+2}^{A,m-1} \, , \\
\mathcal{F}_{m,m}^{3,k} &:= \varepsilon_B^A \{ \mathcal{R}_{B,1}^{A,2k} R_{a_{2k+1} a_{2k+2} b_{2k+2} m} \mathcal{L}_{B,2k+3}^{A,m-1} \}_{:b_{2k+1}} \, .
\end{aligned} \tag{1.8.l}$$

By definition, the empty product is 1. Consequently if $t < s$, then we shall set $\mathcal{R}_{B,s}^{A,t} = 1$ and $\mathcal{L}_{B,s}^{A,t} = 1$. We shall also set $\mathcal{R}_{B,s}^{A,t} = 1$ if $t - s$ is even. Thus

$$\mathcal{F}_{m,m}^{3,0} = \begin{cases} \varepsilon_B^A R_{a_1 a_2 m b_1 : b_2} & \text{if } m = 3 \, , \\ \varepsilon_B^A (R_{a_1 a_2 m b_1} \mathcal{L}_{B,3}^{A,m-1})_{:b_2} & \text{if } m \geq 4 \, . \end{cases}$$

We extend previous results for the interior invariants to this setting as:

Lemma 1.8.10

1. *We have* $\tilde{\mathcal{Q}}_{n,m} \cap \ker r = \{0\}$ *if* $n < m - 1$.
2. *We have* $\tilde{\mathcal{Q}}_{m-1,m}^{\partial M} \cap \ker r = \operatorname{Span}_k \{\mathcal{F}_{m-1,m}^{k}\}$.

3. We have $\tilde{\mathcal{Q}}_{m,m} \cap \ker r = \operatorname{Span}_k\{\mathcal{F}^{1,k}_{m,m}, \mathcal{F}^{2,k}_{m,m}\} + \{\tilde{\mathcal{P}}^{\partial M}_{m,m} \cap \ker r\}$.

4. If $P \in \tilde{\mathcal{P}}^{\partial M}_{m,m} \cap \ker r$ *and if* $\mathfrak{I}(P) = 0$, *then* $P \in \operatorname{Span}_k\{\mathcal{F}^{3,k}_{m,m}\}$.

Proof: Let $0 \neq Q \in \tilde{\mathcal{Q}}_{n,m} \cap \ker r$. Let A be a monomial of Q of weight n for

$$A := \phi_{;\alpha_1} \cdots \phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1;\beta_1} \cdots R_{i_v j_v k_v \ell_v;\beta_v} L_{a_1 b_1:\gamma_1} \cdots L_{a_w b_w:\gamma_w},$$
$$n := \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (|\beta_\nu| + 2) + \sum_{\sigma=1}^{w} (|\gamma_\sigma| + 1).$$

To ensure that $rQ = 0$, we contract $2(m-1)$ tangential indices in A using the ε tensor; the remaining tangential indices must be contracted in pairs. Since the structure group is $O(m-1)$, the normal index "m" can stand alone and unchanged. Let k_T be the total number of tangential indices in A, and let k_m be the total number of times the normal index m appears in A. Then

$$\begin{aligned} 2m-2 &\leq k_T \leq k_T + k_m \\ &= \sum_{\mu=1}^{u} |\alpha_\mu| + \sum_{\nu=1}^{v} (|\beta_\nu| + 4) + \sum_{\sigma=1}^{w} (|\gamma_\sigma| + 2) \\ &= n + 2v + w \\ &= 2n - \sum_{\mu=1}^{u} |\alpha_\mu| - \sum_{\nu=1}^{v} |\beta_\nu| - \sum_{\sigma=1}^{w} |\gamma_\sigma| \leq 2n. \end{aligned} \tag{1.8.m}$$

Assertion (1) of the Lemma follows as this is not possible if $n < m-1$.

We set $n = m-1$ to prove Assertion (2). All the inequalities of Display (1.8.m) must have been equalities so there are no covariant derivatives and thus the ϕ variables do not appear. All the indices are tangential and are contracted using the ε tensor. After using the first Bianchi identity, we see that this leads to the invariants $\mathcal{F}^k_{m-1,m}$ which proves Assertion (2).

Let $n = m$. Display (1.8.m) involves a total increase of 2. Thus at most 2 explicit covariant derivatives are present. However, unless at least 2 covariant derivatives are present, ϕ is not involved and this leads to invariants which belong to $\tilde{\mathcal{P}}_{m,m} \cap \ker r$. Thus we may suppose exactly 2 explicit covariant derivatives are present, and that all of them appear on ϕ. Consequently

$$k_T = 2m-2, \quad k_m = 0, \quad \sum_{\mu=1}^{u} |\alpha_\mu| = 2,$$
$$\sum_{\nu=1}^{v} |\beta_\nu| = 0, \quad \text{and} \quad \sum_{\sigma=1}^{w} |\gamma_\sigma| = 0.$$

Since every index is tangential and all are contracted using the tensor ε, after applying the Bianchi identities, we obtain the invariants $\mathcal{F}^{1,k}_{m,m}$ and $\mathcal{F}^{2,k}_{m,m}$. This completes the proof of Assertion (3).

To prove Assertion (4), we set $\phi = 0$ and consider only metric invariants. Let $\tilde{\mathcal{P}}^p_{n,m}$ be the space of p form valued invariants which are homogeneous of degree n in the derivatives of the metric; $\tilde{\mathcal{P}}_{n,m} = \tilde{\mathcal{P}}^0_{n,m}$. Let

$$\tilde{\delta} : \tilde{\mathcal{P}}^p_{n,m} \to \tilde{\mathcal{P}}^{p-1}_{n+1,m}$$

be induced by the coderivative of the boundary. Lemma 1.8.6 extends to this setting to become

1. r is a surjective map from $\tilde{\mathcal{P}}^p_{n,m}$ to $\tilde{\mathcal{P}}^p_{n,m-1}$ with $r\tilde{\delta}^{\partial M} = \tilde{\delta}^{\partial N} r$.
2. If $n \neq m-1$, then $\tilde{\mathcal{P}}^0_{n,m} \cap \ker \mathfrak{I} = \tilde{\delta}\tilde{\mathcal{P}}^1_{n-1,m}$.
3. If $n \neq m-1$, then $\tilde{\mathcal{P}}^1_{n-1,m} \cap \ker \tilde{\delta} = \tilde{\delta}\tilde{\mathcal{P}}^2_{n-2,m}$.

Let $P_{m,m} \in \tilde{\mathcal{P}}_{m,m} \cap \ker r \cap \ker \mathfrak{I}$. Choose

$$P^1_{m-1,m} \in \tilde{\mathcal{P}}^1_{m-1,m} \quad \text{so} \quad \tilde{\delta} P^1_{m-1,m} = P_{m,m}\,.$$

Unfortunately, $rP^1_{m-1,m}$ need not vanish and we must adjust $P^1_{m-1,m}$ in exactly the same fashion as was done previously. Since

$$\tilde{\delta} r P^1_{m-1,m} = r\tilde{\delta} P^1_{m-1,m} = rP_{m,m} = 0\,,$$

we may choose

$$P^2_{m-2,m-1} \in \tilde{\mathcal{P}}^2_{m-2,m-1} \quad \text{so} \quad \tilde{\delta} P^2_{m-2,m-1} = rP^1_{m-1,m}\,.$$

Since r is surjective, we may choose

$$P^2_{m-2,m} \in \tilde{\mathcal{P}}^2_{m-2,m} \quad \text{so} \quad rP^2_{m-2,m} = P^2_{m-2,m-1}\,.$$

We then have

$$\begin{aligned}
\tilde{\delta}\{P^1_{m-1,m} - \tilde{\delta} P^2_{m-2,m}\} &= \tilde{\delta} P^1_{m-1,m} = P_{m,m},\\
r\{P^1_{m-1,m} - \tilde{\delta} P^2_{m-2,m}\} &= rP^1_{m-1,m} - \tilde{\delta} r P^2_{m-2,m}\\
&= rP^1_{m-1,m} - \tilde{\delta} P^2_{m-2,m-1} = 0\,.
\end{aligned}$$

Consequently

$$\tilde{\mathcal{P}}_{m,m} \cap \ker r \cap \ker \mathfrak{I} = \tilde{\delta}\{\tilde{\mathcal{P}}^1_{m-1,m} \cap \ker r\}\,. \tag{1.8.n}$$

We clear the previous notation. Let $0 \neq P^1_{m-1,m} \in \tilde{\mathcal{P}}^1_{m-1,m} \cap \ker r$ and let

$$A = R_{i_1 j_1 k_1 \ell_1;\beta_1} \ldots R_{i_v j_v k_v \ell_v;\beta_v} L_{a_1 b_1:\gamma_1} \ldots L_{a_w b_w:\gamma_w} e^c$$

be a monomial of $P^1_{m-1,m}$. Since $rP^1_{m-1,m} = 0$, we must contract $2(m-1)$ indices in A using the ε tensor and contract the remaining indices in pairs. We estimate

$$\begin{aligned}
2(m-1) \le k_T \le k_T + k_m &= \sum_{\nu=1}^{v}(|\beta_\nu| + 4) + \sum_{\sigma=1}^{w}(|\gamma_\sigma| + 2) + 1\\
&= n + 2v + w + 1 = 2n - \sum_{\nu=1}^{v}|\beta_\nu| - \sum_{\sigma=1}^{w}|\gamma_\sigma| + 1 \le 2n + 1\,.
\end{aligned} \tag{1.8.o}$$

As $n = m-1$, this sequence of inequalities represents a total increase of 1.

Thus $k_T = 2(m-1)$ and every tangential index is contracted using the ε tensor. By Lemma 1.1.4,

$$L_{c_2c_3:c_1} - L_{c_1c_3:c_2} = R_{c_1c_2c_3m} .$$

This permits us to assume $|\gamma_\sigma| = 0$ and consequently there are no tangential derivatives of L present. If $k_m = 0$, then every index is contracted using the ε tensor. Thus the Bianchi identities show $|\beta_\nu| = 0$ for all ν. This means that every inequality in Display (1.8.o) is an equality which is impossible. Consequently $k_m = 1$ and $\sum_\nu |\beta_\nu| = 0$. This leads to the invariants

$$\mathcal{G}^k_{m-1,m} := \varepsilon^B_A \mathcal{R}^{A,2k}_{B,1} R_{a_{2k+1}a_{2k+2}mb_{2k+1}} \mathcal{L}^{A,m-1}_{B,2k+3} e^{b_{2k+2}} .$$

Assertion (4) now follows from Equation (1.8.n) as

$$\tilde\delta \mathcal{G}^k_{m-1,m} = -\mathcal{F}^{3,k}_{m,m} .$$

This completes the proof. □

The following Lemma is now an immediate consequence of Lemma 1.8.10, Equation (1.8.j), and Equation (1.8.k). We adopt the notation of Display (1.8.l).

Lemma 1.8.11 *There exist constants so that:*

1. $a^{d+\delta}_{m,m,0} = \sum_k c^k_{m,m,0} \mathcal{F}^k_{m-1,m}$.
2. $a^{d+\delta}_{m+1,m,1} = \sum_k c^k_{m+1,m,1} \mathcal{F}^k_{m-1,m}$.
3. $a^{d+\delta}_{m+1,m,0} = \sum_{i,k} c^{i,k}_{m+1,m,0} \mathcal{F}^{i,k}_{m,m}$.

1.9 Chern-Gauss-Bonnet Theorem

We adopt the notational conventions established in Section 1.8. Let M be a compact Riemannian manifold of dimension m with smooth boundary and let $\mathcal{B}_a$ define absolute boundary conditions. We set $\phi = 0$ to define

$$a_{n,m}(g)(x) := \sum_{p=0}^m (-1)^p a_n(x, \Delta^p) \quad \text{and}$$
$$a_{n,m,k}(g)(y) := \sum_{p=0}^m (-1)^p a_{n,k}(y, \Delta^p, \mathcal{B}_a) .$$

If M is closed, then the following result follows from the pioneering work of Patodi [301]; it had been conjectured by McKean and Singer [278] who had established it in the special case that $m = 2$. Subsequently, additional proofs were given by Atiyah, Bott, and Patodi [7] and by Gilkey [173]. The extension to manifolds with boundary was established later in [175]. We will discuss similar results subsequently in Section 3.8 for the Witten Laplacian if $\phi \neq 0$.

Theorem 1.9.1 *Adopt the notation given above.*

1. *If* $m = 2\bar{m}$ *is even, then* $a^{d+\delta}_{m,m}(g) = \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} \varepsilon^I_J \mathcal{R}^{I,m}_{J,1}$.
2. *If* $m = 2\bar{m} + 1$ *is odd, then* $a^{d+\delta}_{m,m}(g) = 0$.
3. *Impose absolute boundary conditions. Then*

$$a^{d+\delta}_{m,m,0}(\phi, g) = \sum_k \frac{1}{\pi^k 8^k k!(m-1-2k)!\mathrm{vol}(\mathrm{S}^{m-1-2k})} \varepsilon^A_B \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1}.$$

We can use Lemma 1.5.10 to prove the *Chern-Gauss-Bonnet Theorem* [122, 123]; this result expresses the Euler-Poincaré characteristic $\chi(M)$ in terms of curvature.

Theorem 1.9.2 *Let* M *be a compact Riemannian manifold which has smooth boundary* ∂M.

1. *If* $m = 2\bar{m}$ *is even, then*

$$\begin{aligned}\chi(M) &= \int_M \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} \varepsilon^I_J \mathcal{R}^{I,m}_{J,1} \\ &+ \int_{\partial M} \sum_k \frac{1}{\pi^k 8^k k!(m-1-2k)!\mathrm{vol}(\mathrm{S}^{m-1-2k})} \varepsilon^A_B \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1}.\end{aligned}$$

2. *If* $m = 2\bar{m} + 1$ *is odd, then*

$$\chi(M) = \int_{\partial M} \sum_k \frac{1}{\pi^k 8^k k!(m-1-2k)!\mathrm{vol}(\mathrm{S}^{m-1-2k})} \varepsilon^A_B \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1}.$$

Subsequently, the local index density for the classical elliptic complexes was identified by Atiyah, Bott, and Patodi [7] who used this to give a heat equation proof of the Atiyah-Singer index theorem in complete generality. The case of manifolds with boundary was studied in [175]. We also refer to [61, 63, 189, 279] for other treatments of the local index theorem.

Patodi's approach involved a direct calculation analyzing cancellation formulae for the fundamental solution of the heat equation. Atiyah, Bott, and Patodi used invariance theory to identify the local index density for the twisted signature and twisted spin complexes. By expressing the de Rham complex, at least locally, in terms of the twisted spin complex, they gave a second proof of Patodi's result for the de Rham complex.

Proof: We first analyze the interior invariants. Suppose $m = 2\bar{m}$. We set $\phi = 0$ and apply Lemma 1.8.9 to see there is a universal constant $c_{m,m}$ so

$$a^{d+\delta}_{m,m}(g)(x) = c_{m,m} \varepsilon^I_J \mathcal{R}^{I,m}_{J,1}.$$

Let $M_m := S^2 \times ... \times S^2$ with the standard metric. We then have that

$$\begin{aligned}2^{\bar{m}} = \chi(M_m) &= \int_{M_m} a^{d+\delta}_{m,m}(g)(x)dx \\ &= 4^{\bar{m}} \bar{m}! c_{m,m} \mathrm{vol}\,(M_m) = 4^{\bar{m}} \bar{m}! c_{m,m} (4\pi)^{\bar{m}}.\end{aligned}$$

We use this relation to establish Assertion (1) of Theorem 1.9.1.

Assertion (2) of Theorem 1.9.1 follows from Lemma 1.8.9.

We apply Lemma 1.8.11 to express

$$a_{m,m,0} = \sum_k c^k_{m,m,0} \varepsilon^A_B \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1} \,.$$

We let $M_{m,k} := S^2 \times ... \times S^2 \times D^{m-2k}$ where $0 \le 2k \le m-2$. Then

$$\begin{array}{rcl} 2^k & = & \chi(M_{m,k}) \\ & = & 4^k k!(m-1-2k)! c^k_{m,m,0} \operatorname{vol}(\partial M_{m,k}) \\ & = & (16\pi)^k k!(m-1-2k)! \operatorname{vol}(S^{m-1-2k}) \,. \end{array}$$

We solve this equation for $c^k_{m,m,0}$ to complete the proof of Theorem 1.9.1. $\square$

Chapter 2

Heat Content Asymptotics

2.0 Introduction

In Chapter 2, we discuss the heat content asymptotics. We adopt the following notational conventions throughout. Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M. Let V be a smooth vector bundle over M and let D be an operator of Laplace type on $C^\infty(V)$. Let $D_\mathcal{B}$ be the realization of D with respect to the boundary conditions defined by a suitable boundary operator $\mathcal{B}$. We shall always assume that $(D, \mathcal{B})$ is elliptic with respect to a cone $\mathcal{C}_\delta$ for $0 \le \delta < \frac{\pi}{2}$; we refer to the discussion in Sections 1.4 and 1.5 for further details. Let $D_\mathcal{B}$ be the associated realization.

Let $\tilde{D}$ be the adjoint operator on the dual bundle V^* and let $\tilde{\mathcal{B}}$ be the adjoint boundary operator; the operator $\tilde{B}$ will be an operator of the same type which is defined by the dual structures on V^* and hence $(\tilde{D}, \tilde{\mathcal{B}})$ will also be elliptic with respect to the cone $\mathcal{C}_\delta$. Let ∇ and E be the canonical connection and endomorphism determined by D as discussed in Lemma 1.2.1. Let $\tilde{\nabla}$ and $\tilde{E}$ be the dual structures on V^*. By Lemma 1.2.2,

$$D\phi = -(\phi_{;ii} + E\phi) \quad \text{and} \quad \tilde{D} = -(\rho_{;ii} + \tilde{E}\rho)\,.$$

Let $\phi \in C^\infty(V)$ be the initial temperature distribution. Let $u = e^{-tD_\mathcal{B}}\phi$ be the subsequent temperature; u is characterized by Display (1.4.d)

$$\begin{array}{ll} (\partial_t + D)u = 0 \quad \text{for} \quad t > 0 & \text{(evolution equation)} \\ \mathcal{B}u = 0 \quad \text{for} \quad t > 0 & \text{(boundary condition)} \\ u|_{t=0} = \phi & \text{(initial condition).} \end{array}$$

The initial condition is to be understood in the distributional sense if ∂M is non-empty. This means that

$$\lim_{t \downarrow 0} \int_M \langle u(x;t), \rho(x) \rangle dx = \int_M \langle \phi(x), \rho(x) \rangle dx \quad \text{for all} \quad \rho \in C^\infty(V^*)\,.$$

Let $\rho \in C^\infty(V^*)$ be the *specific heat*. The *total heat energy content* of the manifold is defined to be

$$\beta(\phi, \rho, D, \mathcal{B})(t) := \int_M \langle u(x;t), \rho(x)\rangle dx\,.$$

By Theorem 1.4.7, there is a complete asymptotic expansion as $t \downarrow 0$ of the form

$$\beta(\phi, \rho, D, \mathcal{B})(t) \sim \sum_{n=0}^{\infty} t^{n/2} \beta_n(\phi, \rho, D, \mathcal{B}),$$

where the *heat content asymptotic coefficients* β_n are locally computable

$$\beta_n(\phi, \rho, D, \mathcal{B}) \quad = \quad \int_M \beta_n^M(\phi, \rho, D)(x)dx + \int_{\partial M} \beta_n^{\partial M}(\phi, \rho, D, \mathcal{B})(y)dy\,.$$

The interior invariants are completely determined by Equation (1.4.f);

$$\beta_n^M(\phi, \rho, D)(x) = \begin{cases} 0 & \text{if} \quad n \text{ is odd,} \\ (-1)^k \frac{1}{k!}\langle D^k\phi, \rho\rangle(x) & \text{if} \quad n = 2k\,. \end{cases}$$

Thus we shall focus on computing the boundary integrands for various boundary conditions. We will often replace the interior integrand $\frac{1}{2}\langle D^2\phi, \rho\rangle$ by the more symmetric integrand $\frac{1}{2}\langle D\phi, \tilde{D}\rho\rangle$ in discussing β_4. This requires changing the formula for $\beta_4^{\partial M}$ to include the appropriate correction term in the Green's formula. This change is motivated by preserving the symmetry, established in Lemma 2.1.3, between the roles of ϕ and ρ

$$\beta_n(\phi, \rho, D, \mathcal{B}) = \beta_n(\rho, \phi, \tilde{D}, \tilde{\mathcal{B}})\,.$$

In Sections 2.1 and 2.2 we review the functorial properties which these invariants have for quite general boundary conditions. There are many functorial properties which are independent of the particular boundary condition studied. The symmetry property of Lemma 2.1.3 and the product formula of Lemmas 2.1.8, 2.1.9, and 2.1.10 are true quite generally. There are also relations between the heat content asymptotics for various boundary conditions. For example Lemma 2.1.15 relates heat content asymptotics for Dirichlet and Robin boundary conditions, Lemma 2.2.1 and Lemma 2.2.5 relate transmission and transfer boundary conditions to Dirichlet and Robin boundary conditions, and Lemma 2.2.16 relates heat content asymptotics for mixed and spectral boundary conditions. There are also many results which are specific to the particular boundary condition studied.

In subsequent sections, we treat various boundary conditions separately. In Section 2.3 we study the heat content asymptotics for Dirichlet boundary conditions. In Theorems 2.3.1 and 2.3.2, we discuss the heat content asymptotics in the special case that $\phi = \rho = 1$ and D is the scalar Laplacian for the unit disk or the upper hemisphere, respectively. In Theorem 2.3.3, we determine $\beta_n(\phi, \rho, D, \mathcal{B})$ for general ϕ and ρ and for general operators of Laplace type for $n \leq 4$. In Theorem 2.3.4, we return to the special case in which D is the scalar Laplacian and $\phi = \rho = 1$ to give β_5. We shall follow the discussion in

[46] for the proof of Theorems 2.3.1 and 2.3.3; we shall not prove Theorems 2.3.2 or 2.3.4 but instead refer to [37, 47] and [49], respectively, for further details.

In Section 2.4, we discuss the heat content asymptotics for Robin boundary conditions; these boundary conditions are defined by the operator

$$\mathcal{B}\phi := (\phi_{;m} + S\phi)|_{\partial M}$$

where $\phi_{;m}$ denotes the covariant derivative of ϕ with respect to the inward unit normal vector field and where S is an auxiliary endomorphism of $V|_{\partial M}$. In Theorem 2.4.1, we follow the discussion of [45, 132] to give the heat content asymptotics for $n \leq 6$. The results of Section 2.3 play an important role in this discussion as Lemma 2.1.15 relates the Dirichlet and the Robin heat content asymptotics. In Section 2.5, we combine the results of Sections 2.3 and 2.4 to treat the invariants $\beta_n(\phi, \rho, D, \mathcal{B})$ where $\mathcal{B}$ is the mixed boundary operator

$$\mathcal{B}\phi := \Pi_+(\phi_{;m} + S\phi)|_{\partial M} \oplus \Pi_-\phi|_{\partial M}$$

introduced in Section 1.5.3. In addition to terms that arise from decoupling the problem as a direct sum of Dirichlet and Robin boundary conditions, there are new contributions which reflect the failure of the problem to decouple. These terms arise both from the endomorphism E which can intertwine the Dirichlet and Neumann bundles V_- and V_+ as well as from the failure of the splitting $V|_{\partial M} = V_+ \oplus V_-$ to be parallel with respect to the connection ∇. In Theorem 2.5.1, we present results of [132] giving the heat content asymptotics in this setting for $n \leq 3$. In principle, β_4 could be computed, but the essential new features are already manifest at the β_3 level.

In Sections 2.6 and 2.7, we follow the discussion in [194] to discuss transmission and transfer boundary conditions. Much of the computation relies upon the functorial properties which relate the heat content asymptotics with these boundary conditions to the corresponding heat content asymptotics with Dirichlet and Robin boundary conditions. There are, however, significant new phenomena which relate to the singular structures involved and which relate to the results in Lemmas 2.2.2 and 2.2.3 discussed in Section 2.2. We give in Theorems 2.6.1 and 2.7.1 the heat content asymptotics for $n \leq 3$ for transmission boundary conditions and for transfer boundary conditions, respectively.

In Section 2.8, we study oblique boundary conditions

$$\mathcal{B}\phi := (\nabla_{e_m} + \mathcal{B}_T)\phi|_{\partial M}$$

where $\mathcal{B}_T$ is a suitably chosen tangential operator. The ellipticity condition is not automatic in this setting, but it does follow for small tangential operators by Lemma 1.6.8. It is somewhat surprising that the heat content asymptotics in this setting are not sensitive to the loss of ellipticity; the formulae of Theorem 2.8.1, which summarizes work of [196], remains valid for all tangential operators $\mathcal{B}_T$. By contrast, the formulae for the heat trace asymptotics given in [25, 146] become singular when the ellipticity condition breaks down.

In Section 2.9, we study time-dependent phenomena where the operators D_t

in question are operators of Laplace type for a smooth 1 parameter family of time-dependent Riemannian metrics. We consider either Dirichlet boundary conditions or a smooth 1 parameter family of Robin boundary conditions. Theorem 2.9.1 generalizes results of [190] and computes β_n for $n \le 4$ in this setting.

In Section 2.10, we return to the discussion of static operators; these are operators whose coefficients are independent of t. We follow the discussion of [49, 51] and consider the inhomogeneous problem

$$\begin{array}{lll} (\partial_t + D)u(x;t) = p(x;t) & t > 0 & \text{(evolution equation)}, \\ \mathcal{B}u(y;t) = \psi(y;t) & t > 0, y \in \partial M & \text{(boundary condition)}, \\ u|_{t=0} = \phi & & \text{(initial condition)}. \end{array}$$

Here p is a smooth time-dependent potential function describing heat sources and sinks in the interior and ψ is a smooth time-dependent function giving the heat flow on the boundary. The problems decouple; we can consider the effect of p, of ψ, and of a time-dependent specific heat ρ separately. In Theorem 2.10.1, we deal with time-dependent specific heats $\rho(x;t)$ and in Theorem 2.10.3 we study the potential $p(x;t)$. Both theorems relate the resulting heat content asymptotics to previously computed invariants. The most interesting phenomena relate to the heat transfer defined by ψ. Expand

$$\psi \sim \sum_{i=0}^{\infty} t^i \psi_i$$

in a Taylor series. Assuming one can find harmonic functions h_i so $\mathcal{B}h_i = \psi_i$, Theorem 2.10.6 relates the heat content asymptotics defined by the boundary heat pump ψ to the heat content asymptotics with initial conditions h_i. In Theorem 2.10.10, we show that the heat content asymptotics $\beta_n(h_i, \rho, D, \mathcal{B})$ for $n \ge 1$ are in fact determined by ψ_i, and thus the formulae of Theorem 2.10.6 are local in ψ_i. In the proof of Theorem 2.10.9, we add a fictitious Dirichlet boundary to facilitate finding such harmonic functions h_i and thereby solve the problem of a boundary heat pump without any additional assumptions.

In Section 2.11 we discuss non-minimal operators of the form

$$D := Ad\delta + B\delta d - E \quad \text{on} \quad C^\infty(T^*M).$$

These are not operators of Laplace type. We follow the discussion of [188] to determine the invariants β_0, β_1, and β_2 in this setting for relative and absolute boundary conditions.

In Section 2.12, we generalize the discussion in [197] to present some preliminary work concerning the invariants β_0, β_1, and β_2 for spectral boundary conditions.

In our study of the boundary invariants for various boundary conditions, we shall have to introduce various universal constants. We clear the previous notation in each new Section; thus the constants c_i of Section 2.3 have no relation to the constants of Section 2.4.

2.1 Functorial properties I

The heat content asymptotics have a number of functorial or naturality properties that will be essential to our study of these invariants. Until much later in this section, we work in great generality. We suppose M is a compact Riemannian manifold with smooth boundary ∂M. We fix $0 \le \delta < \frac{\pi}{2}$; we shall take $\delta = 0$ except when considering oblique boundary conditions.

Let D be a second order partial differential operator and let $\mathcal{B}$ be a boundary operator. Let $(\tilde{D}, \tilde{\mathcal{B}})$ be the dual structures on V^*. We say that $(D, \mathcal{B})$ is *admissible* if $(D, \mathcal{B})$ and $(\tilde{D}, \tilde{\mathcal{B}})$ are elliptic with respect to the cone $\mathcal{C}_\delta$ for some $0 \le \delta < \frac{\pi}{2}$. If D is an operator of Laplace type and if $\mathcal{B}$ is one of the boundary operators discussed in Sections 1.5 and 1.6, then $(D, \mathcal{B})$ is admissible.

We shall implicitly assume a linkage between the operator D and the Riemann metric. If D is an operator of Laplace type, then the leading symbol of D is given by the metric tensor. Similarly, if $D = Ad\delta + B\delta d - E$ is a *non-minimal operator* on $C^\infty(T^*M)$, then again the linkage is clear. In the general setting, the linkage must be imposed externally to make sense of the results concerning product formulae and dimensional analysis given in Sections 2.1.7 and 2.1.8; we shall not bother to discuss the nature of this linkage in the general setting as the two examples cited above are the only ones we shall be interested in.

It is sometimes not safe to commute a spectral invariant function, such as the heat content asymptotics or the heat trace asymptotics, with a variation. For example, the eta invariant is not continuous as an $\mathbb{R}$ valued invariant, although it is continuous when regarded as a $\mathbb{R}/\mathbb{Z}$ valued invariant. However as we shall be dealing for the most part with invariants which are given by a local formula, the methods described in [189] Lemma 1.9.3 (d) apply and can be extended to the class of manifolds with boundary as long as the boundary condition is not varied. We shall therefore not belabor the point unduly but shall simply commute the two processes as needed henceforth.

In Section 2.9, we will study variable geometries. Let $\mathfrak{D} = \{D_t\}$ be a smooth 1 parameter family of second order operators and let $\mathfrak{B} = \{\mathcal{B}_t\}$ be a smooth 1 parameter family of boundary operators. We say that $(\mathfrak{D}, \mathfrak{B})$ is *admissible* if $(D_t, \mathcal{B}_t)$ is admissible for all t in the parameter range. Many, but not all, of our results hold true in this slightly greater generality. We shall formally let $u = e^{-t\mathfrak{D}_\mathfrak{B}}\phi$ denote the solution to Equation (1.6.p). This is **not** defined by the functional calculus.

2.1.1 The interior integrands

If $(D, \mathcal{B})$ is admissible, then the heat content asymptotics are well defined. The following observation will be fundamental to our study.

Lemma 2.1.1 *If $(D, \mathcal{B})$ is admissible, then:*

1. $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle dx$.

2. If $k \geq 1$, *then we may take* $\beta^M_{2k-1}(\phi,\rho,D,\mathcal{B})(x) = 0$.

3. If $k \geq 1$, *then we may take* $\beta^M_{2k}(\phi,\rho,D,\mathcal{B})(x) = (-1)^k\frac{1}{k!}\langle D^k\phi,\rho\rangle(x)$.

Proof: Let $u := e^{-tD_{\mathcal{B}}}\phi$. Since $\lim_{t\downarrow 0} u(\cdot;t) = \phi$ in the distributional sense,

$$\lim_{t\downarrow 0}\int_M \langle u(x;t),\rho(x)\rangle dx = \int_M \langle\phi(x),\rho(x)\rangle dx\,.$$

Assertion (1) now follows. Furthermore, Assertions (2) and (3) follow from Equation (1.4.f). $\square$

2.1.2 Shifting the spectrum

The following Lemma is useful in studying the dependence of β_n upon the auxiliary endomorphism E.

Lemma 2.1.2 *Let* $(\mathfrak{D},\mathfrak{B})$ *be admissible.*

1. $\beta(\phi,\rho,\mathfrak{D}-\varepsilon\mathrm{Id}\,,\mathfrak{B})(t) = e^{t\varepsilon}\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t)$.

2. $\partial_\varepsilon|_{\varepsilon=0}\beta_n(\phi,\rho,\mathfrak{D}-\varepsilon\mathrm{Id}\,,\mathfrak{B}) = \beta_{n-2}(\phi,\rho,\mathfrak{D},\mathfrak{B})$.

Proof: Let $u = e^{-t\mathfrak{D}_{\mathcal{B}}}\phi$. Let $D_{t,\varepsilon} := D_t - \varepsilon\mathrm{Id}$ define the 1 parameter family $\mathfrak{D}_\varepsilon$. Let $u_\varepsilon(x;t) := e^{t\varepsilon}u(x;t)$. We show that

$$u_\varepsilon = e^{-t\mathfrak{D}_{\varepsilon,\mathcal{B}}}\phi;$$

the assertions of the Lemma will then follow. We compute

$$\begin{aligned}
&(\partial_t + D_{t,\varepsilon})u_\varepsilon = \partial_t u_\varepsilon + D_{t,\varepsilon}u_\varepsilon = e^{t\varepsilon}(\partial_t u + \varepsilon u) + e^{t\varepsilon}(D_t u - \varepsilon u) = 0,\\
&\mathcal{B}_t u_\varepsilon = e^{t\varepsilon}\mathcal{B}_t u = 0,\\
&u_\varepsilon|_{t=0} = u|_{t=0} = \phi. \quad \square
\end{aligned}$$

2.1.3 A duality relationship

We relate the heat content asymptotics of $(D,\mathcal{B})$ and of $(\tilde{D},\tilde{\mathcal{B}})$; this lemma does not generalize to the time-dependent setting.

Lemma 2.1.3 *Let* $(D,\mathcal{B})$ *be admissible. Then:*

1. $\beta(\phi,\rho,D,\mathcal{B})(t) = \beta(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})(t)$.

2. $\beta_n(\phi,\rho,D,\mathcal{B}) = \beta_n(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})$.

Proof: Let $u(x;t) := e^{-tD_{\mathcal{B}}}\phi$ and let $\tilde{u}(x;t) := e^{-t\tilde{D}_{\tilde{\mathcal{B}}}}\rho$. Let $\partial_2 u$ and $\partial_2\tilde{u}$ denote the derivatives of u and of $\tilde{u}$, respectively, with respect to the temporal parameter t. By the defining equation,

$$\partial_2 u = -Du \quad\text{and}\quad \partial_2\tilde{u} = -\tilde{D}\tilde{u}\,.$$

Fix $t > 0$. For $0 \leq s \leq t$, we define

$$f(s) := \int_M \langle u(x;s),\tilde{u}(x;t-s)\rangle dx.$$

We may then use the chain rule and the product rule to compute

$$\begin{aligned}\partial_s f(s) &= \int_M \Big\{\langle \partial_2 u(x;s), \tilde{u}(x;t-s)\rangle - \langle u(x;s), \partial_2\tilde{u}(x;t-s)\rangle\Big\}dx \\ &= \int_M \Big\{-\langle Du(x;s), \tilde{u}(x;t-s)\rangle + \langle u(x;s), \tilde{D}u(x;t-s)\rangle\Big\}dx\,.\end{aligned}$$

Since $\mathcal{B}u = 0$ and $\tilde{\mathcal{B}}\tilde{u} = 0$, the boundary terms vanish in the Green's formula and consequently we have that

$$\int_M \Big\{-\langle Du(x;s), \tilde{u}(x;t-s)\rangle + \langle u(x;s), \tilde{D}u(x;t-s)\rangle\Big\}dx = 0.$$

This implies $\partial_s f(s) = 0$ so $f(s)$ is constant. Thus

$$\begin{aligned}0 &= f(t) - f(0) = \int_M \Big\{\langle u(x;t), \tilde{u}(x;0)\rangle - \langle u(x;0), \tilde{u}(x;t)\rangle\Big\}dx \\ &= \int_M \Big\{\langle u(x;t), \rho(x)\rangle - \langle \phi(x), \tilde{u}(x;t)\rangle\Big\}dx \\ &= \beta(\phi,\rho,D,\mathcal{B})(t) - \beta(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})(t)\,.\end{aligned}$$

This establishes the first assertion; the second then follows from the first. □

2.1.4 A recursion relation

We can generalize the recursion relation of Equation (1.3.e) to the case of manifolds with boundary if ϕ satisfies the boundary condition under consideration.

Lemma 2.1.4 *Let $(D,\mathcal{B})$ be admissible. If $\mathcal{B}\phi = 0$, then:*
1. $\partial_t\beta(\phi,\rho,D,\mathcal{B})(t) = -\beta(D\phi,\rho,D,\mathcal{B})(t)$.
2. $\beta_n(\phi,\rho,D,\mathcal{B}) = -\frac{2}{n}\beta_{n-2}(D\phi,\rho,D,\mathcal{B})$.

Proof: We shall use the same argument as that given in the proof of Assertions (2) and (3) of Theorem 1.3.12 to establish Equation (1.3.e). Let

$$\tilde{u}(x;t) = e^{-t\tilde{D}_{\tilde{\mathcal{B}}}}\rho\,.$$

Since $\tilde{\mathcal{B}}\tilde{u} = 0$ and $\mathcal{B}\phi = 0$, we can integrate by parts to conclude

$$\int_M \langle \tilde{D}\tilde{u}(x;t), \phi\rangle dx = \int_M \langle \tilde{u}(x;t), D\phi\rangle dx\,.$$

We apply Lemma 2.1.3 to prove the first assertion by computing

$$\begin{aligned}&\partial_t\beta(\phi,\rho,D,\mathcal{B})(t) = \partial_t\beta(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})(t) \\ =\;& \int_M \langle \partial_t\tilde{u}(x;t), \phi(x)\rangle dx = -\int_M \langle \tilde{D}\tilde{u}(x;t), \phi(x)\rangle dx \\ =\;& -\int_M \langle \tilde{u}(x;t), D\phi(x)\rangle dx = -\beta(\rho,D\phi,\tilde{D},\tilde{\mathcal{B}})(t) \\ =\;& -\beta(D\phi,\rho,D,\mathcal{B})(t)\,.\end{aligned}$$

We equate the coefficients of corresponding powers of t in the asymptotic expansions to derive the second assertion from the first. $\square$

We can draw a useful consequence from Lemmas 2.1.1 and 2.1.4 that we shall need subsequently in Section 2.10.

Lemma 2.1.5 *Let $(D, \mathcal{B})$ be admissible. Normalize the interior integrands for β_n as in Lemma 2.1.1. Then there exist natural partial differential operators $T_{p,n}$ so that*

$$\beta_n^{\partial M}(\phi, \rho, D, \mathcal{B}) = \textstyle\int_{\partial M} \sum_p \langle \mathcal{B} D^p \phi, T_{p,n}\rho\rangle dy\,.$$

Proof: Choose a complementary boundary condition $\mathcal{B}_1$ so the Cauchy data

$$\phi|_{\partial M} \oplus \phi_{;m}|_{\partial M}$$

is determined by $\mathcal{B}\phi \oplus \mathcal{B}_1\phi$. For example, if $\mathcal{B}$ defines Dirichlet boundary conditions, then we can let $\mathcal{B}_1$ be the Robin boundary operator for any S; if $\mathcal{B}$ defines Robin boundary conditions, then we can let $\mathcal{B}_1$ be the Dirichlet boundary operator. The Cauchy data of all orders is then specified by

$$\oplus_p \{\mathcal{B} D^p \phi \oplus \mathcal{B}_1 D^p \phi\}\,.$$

We integrate by parts tangentially to see there exist natural partial differential operators $T_{p,n}$ and $U_{p,n}$ so that

$$\beta_n^{\partial M}(\phi, \rho, D, \mathcal{B}) = \int_{\partial M} \left\{ \sum_p \langle \mathcal{B} D^p \phi, T_{p,n}\rho\rangle + \sum_p \langle \mathcal{B}_1 D^p \phi, U_{p,n}\rho\rangle \right\} dy\,.$$

Choose $\Phi \in C^\infty(V)$ so that

$$\begin{aligned} &\mathcal{B} D^p \Phi = 0 \text{ for } 2p \le n, \quad \text{and} \\ &\mathcal{B}_1 D^p \Phi = \mathcal{B}_1 D^p \phi \text{ for } 2p \le n\,. \end{aligned}$$

By Lemmas 2.1.1 and 2.1.4,

$$\beta_n(\Phi, \rho, D, \mathcal{B}) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (-1)^k \frac{1}{k!} \int_M \langle D^k \Phi, \rho\rangle dx & \text{if } n \text{ is even}\,. \end{cases}$$

It now follows that $\beta_n^{\partial M}(\Phi, \rho, D, \mathcal{B}) = 0$ and hence

$$0 = \int_{\partial M} \left\{ \sum_p \langle \mathcal{B}_1 D^p \Phi, U_{p,n}\rho\rangle \right\} dy\,.$$

The Lemma now follows as

$$\int_{\partial M} \left\{ \sum_p \langle \mathcal{B}_1 D^p \phi, U_{p,n}\rho\rangle \right\} dy = \int_{\partial M} \left\{ \sum_p \langle \mathcal{B}_1 D^p \Phi, U_{p,n}\rho\rangle \right\} dy\,. \quad \square$$

2.1.5 Heat content asymptotics for self-adjoint operators

Assume that V is equipped with a positive definite Hermitian inner product $(\cdot,\cdot)$. Set $u := e^{-tD_{\mathcal{B}}}\phi$. If $\rho \in C^\infty(V)$, then we define

$$\beta(\phi,\rho,D,\mathcal{B})(t) := \int_M (u(x;t),\rho(x))dx\,.$$

Note that this is conjugate linear in ρ.

We assume further that $D_{\mathcal{B}}$ is self-adjoint. Let $\{\phi_i,\lambda_i\}$ be a discrete spectral resolution of $D_{\mathcal{B}}$ as discussed in Theorem 1.4.18. We define the *associated Fourier coefficients* for $\phi \in C^\infty(V)$ by setting

$$\sigma_i(\phi) = (\phi,\phi_i)_{L^2} := \int_M (\phi(x),\phi_i(x))dx\,.$$

Lemma 2.1.6 *Let $(D,\mathcal{B})$ be admissible. If $D_{\mathcal{B}}$ is self-adjoint, then:*

1. $e^{-tD_{\mathcal{B}}}\phi = \sum_i e^{-t\lambda_i}\sigma_i(\phi)\phi_i$.
2. $\beta(\phi,\rho,D,\mathcal{B})(t) = \sum_i e^{-t\lambda_i}\sigma_i(\phi)\bar{\sigma}_i(\rho)$.

Proof: The Fourier coefficients $\sigma_i(\phi)$ are uniformly bounded since

$$\sum_i \sigma_i(\phi)^2 = |\phi|^2_{L^2} < \infty.$$

The estimates of Theorem 1.4.18 show the series

$$u(x;t) := \sum_i e^{-t\lambda_i}\sigma_i(\phi)\phi_i.$$

converges uniformly in the C^k norm for any k for $t>0$; it also converges in the L^2 norm for $t=0$. Assertion (1) is now immediate since u satisfies the defining relations given in Display (1.4.d):

$$\begin{aligned}
(\partial_t + D)u &= \sum_i \sigma_i(\phi)(\partial_t + D)e^{-t\lambda_i}\phi_i\\
&= \sum_i \sigma_i(\phi)(-\lambda_i+\lambda_i)e^{-t\lambda_i}\phi_i = 0,\\
\mathcal{B}u &= \sum_i \sigma_i(\phi)e^{-t\lambda_i}\mathcal{B}\phi_i = 0,\\
u(\cdot;0) &= \sum_i \sigma_i(\phi)\phi_i = \phi \quad \text{in} \quad L^2\,.
\end{aligned}$$

The series in question converge uniformly. Consequently, we can interchange summation and integration to prove Assertion (2) by computing

$$\beta(\phi,\rho,D,\mathcal{B})(t) = \int_M (u(x;t),\rho(x))dx$$

$$= \int_M \sum_i e^{-t\lambda_i}\sigma_i(\phi)(\phi_i(x),\rho(x))dx$$

$$\begin{aligned} &= \sum_i e^{-t\lambda_i}\sigma_i(\phi)\int_M (\phi_i(x),\rho(x))dx \\ &= \sum_i e^{-t\lambda_i}\sigma_i(\phi)\bar\sigma_i(\rho)\,. \qquad \square \end{aligned}$$

We use Lemma 2.1.6 to see

$$\begin{aligned} &\beta(\phi,\rho,D,\mathcal{B})(t) = \sum_i e^{-t\lambda_i}\sigma_i(\phi)\bar\sigma_i(\rho), \\ &\beta(\rho,\phi,D,\mathcal{B})(t) = \sum_i e^{-t\lambda_i}\sigma_i(\rho)\bar\sigma_i(\phi), \quad \text{so} \\ &\beta(\phi,\rho,D,\mathcal{B})(t) = \bar\beta(\rho,\phi,D,\mathcal{B})(t)\,. \end{aligned}$$

Thus we recover Lemma 2.1.3 in this setting; the complex conjugate is introduced as the identification of V with V^* is conjugate linear. Similarly, if $\mathcal{B}\phi = 0$, we may integrate by parts to see

$$\begin{aligned} \sigma_i(D\phi) &= \int_M (D\phi(x),\phi_i(x))dx = \int_M (\phi(x),D\phi_i(x))dx \\ &= \lambda_i\int_M (\phi(x),\phi_i(x)) = \lambda_i\sigma_i(\phi)\,. \end{aligned}$$

Consequently, we may give another derivation of Lemma 2.1.4 in this setting by computing

$$\begin{aligned} &\beta(D\phi,\rho,D,\mathcal{B})(t) = \sum_i e^{-t\lambda_i}\lambda_i\sigma_i(\phi)\bar\sigma_i(\rho) \\ = &\ -\partial_t\sum_i e^{-t\lambda_i}\sigma_i(\phi)\bar\sigma_i(\rho) = -\partial_t\beta(\phi,\rho,D,\mathcal{B})(t)\,. \end{aligned}$$

2.1.6 Direct sums

The heat content asymptotics are additive with respect to direct sums.

Lemma 2.1.7 *For $i = 1,2$, let $(\mathfrak{D}_i,\mathfrak{B}_i)$ be admissible. Let*

$$\mathfrak{D} := \mathfrak{D}_1 \oplus \mathfrak{D}_2 \quad \textit{and} \quad \mathfrak{B} := \mathfrak{B}_1 \oplus \mathfrak{B}_2 \quad \textit{on} \quad C^\infty(V_1 \oplus V_2)\,.$$

Let $\phi := \phi_1 \oplus \phi_2$ and $\rho := \rho_1 \oplus \rho_2$ for $\phi_i \in C^\infty(V_i)$ and $\rho_i \in C^\infty(V_i^*)$. Then:
1. $\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t) = \beta(\phi_1,\rho_1,\mathfrak{D}_1,\mathfrak{B}_1)(t) + \beta(\phi_2,\rho_2,\mathfrak{D}_2,\mathfrak{B}_2)(t)$.
2. $\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_n(\phi_1,\rho_1,\mathfrak{D}_1,\mathfrak{B}_1) + \beta_n(\phi_2,\rho_2,\mathfrak{D}_2,\mathfrak{B}_2)$.

Proof: It is immediate that $(\mathfrak{D},\mathfrak{B})$ is admissible. Let

$$u(x;t) := u_1(x;t) \oplus u_2(x;t) \quad \text{where} \quad u_i(x;t) := e^{-t(\mathfrak{D}_i)_{\mathfrak{B}_i}}\phi_i\,.$$

We check that $u = e^{-t\mathfrak{D}_{\mathfrak{B}}}\phi$ by verifying that the defining relations are satisfied

$$\begin{aligned} &(\partial_t + D_t)u = (\partial_t + D_{t,1})u_1 \oplus (\partial_t + D_{t,2})u_2 = 0, \\ &\mathcal{B}u = \mathcal{B}_{t,1}u_1 \oplus \mathcal{B}_{t,2}u_2 = 0, \\ &u|_{t=0} = u_1|_{t=0} \oplus u_2|_{t=0} = \phi_1 \oplus \phi_2 = \phi\,. \end{aligned}$$

Assertion (1) now follows from the computation

$$\begin{aligned}
\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t) &= \int_M \langle u(x;t),\rho(x)\rangle dx \\
&= \int_M \Big\{\langle u_1(x;t),\rho_1(x)\rangle + \langle u_2(x;t),\rho_2(x_2)\rangle\Big\} dx \\
&= \beta(\phi_1,\rho_1,\mathfrak{B}_1,\mathfrak{D}_1)(t) + \beta(\phi_2,\rho_2,\mathfrak{D}_2,\mathfrak{B}_2)(t)\,.
\end{aligned}$$

Assertion (2) follows from Assertion (1) by equating terms in the asymptotic expansions. $\square$

2.1.7 Products

There are a number of product formulae that are important to our investigation. We begin by studying *Cartesian product structures*:

Lemma 2.1.8 *For $i = 1, 2$, let $(\mathfrak{D}_i, \mathcal{B}_i)$ be admissible on vector bundles V_i over compact Riemannian manifolds (M_i, g_i). Let M_1 be closed so no boundary condition is needed for $\mathfrak{D}_1$. Let $(M, g) := (M_1, g_1) \times (M_2, g_2)$ be the product Riemannian manifold and let $V := \pi_1^* V_1 \otimes \pi_2^* V_2$ be the tensor product bundle over M. Define $\mathfrak{D}$ and $\mathfrak{B}$ by setting*

$$D_t := D_{t,1} \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes D_{t,2} \quad \textit{and} \quad \mathcal{B}_t := \mathrm{Id} \otimes \mathcal{B}_{t,2} \quad \textit{on} \quad C^\infty(V)\,.$$

Let $\phi(x) := \phi_1(x_1) \otimes \phi_2(x_2)$ and $\rho(x) := \rho_1(x_1) \otimes \rho_2(x_2)$ for $\phi_i \in C^\infty(V_i)$ and $\rho_i \in C^\infty(V_i^*)$ over M_i. Then:

1. *$\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t) = \beta(\phi_1,\rho_1,\mathfrak{D}_1)(t) \cdot \beta(\phi_2,\rho_2,\mathfrak{D}_2,\mathfrak{B}_2)(t)$.*
2. *$\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \sum_{p+q=n} \beta_p(\phi_1,\rho_1,\mathfrak{D}_1)\beta_q(\phi_2,\rho_2,\mathfrak{D}_2,\mathfrak{B}_2)$.*

Proof: It is immediate that $(\mathfrak{D},\mathfrak{B})$ is admissible. Let

$$\begin{aligned}
&u_1(x_1;t) := e^{-t\mathfrak{D}_1}\phi_1, \\
&u_2(x_2;t) := e^{-t(\mathfrak{D}_2)_{\mathfrak{B}_2}}\phi_2, \qquad \text{and} \\
&u(x;t) := u_1(x_1;t) \otimes u_2(x_2;t)\,.
\end{aligned}$$

We show that $u = e^{-t\mathfrak{D}_\mathfrak{B}}\phi$ by checking that the defining relations are satisfied

$$\begin{aligned}
&(\partial_t + D_t)u = (\partial_t + D_{t,1})u_1 \otimes u_2 + u_1 \otimes (\partial_t + D_{t,2})u_2 = 0, \\
&\mathcal{B}_t u = u_1 \otimes \mathcal{B}_{t,2} u_2 = 0, \\
&u|_{t=0} = u_1|_{t=0} \otimes u_2|_{t=0} = \phi_1 \otimes \phi_2 = \phi\,.
\end{aligned}$$

Consequently, we may establish Assertion (1) by computing

$$\begin{aligned}
\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t) &= \int_M \langle u(x;t),\rho(x)\rangle dx \\
&= \int_M \langle u_1(x_1;t),\rho_1(x_1)\rangle \cdot \langle u_2(x_2;t),\rho_2(x_2)\rangle dx_1 \cdot dx_2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{M_1} \langle u_1(x_1;t), \rho_1(x_1)\rangle dx_1 \cdot \int_{M_2} \langle u_2(x_2;t), \rho_2(x_2)\rangle dx_2 \\
&= \beta(\phi_1, \rho_1, \mathfrak{D}_1)(t) \cdot \beta(\phi_2, \rho_2, \mathfrak{D}_2, \mathfrak{B}_2)(t)\,.
\end{aligned}
$$

Assertion (2) follows from Assertion (1) by equating terms in the asymptotic expansions. $\square$

Lemma 2.1.8 can be generalized to study certain warped products:

Lemma 2.1.9 *For $i = 1, 2$, let $(\mathfrak{D}_i, \mathfrak{B}_i)$ be admissible on vector bundles V_i over compact Riemannian manifolds (M_i, g_i) of dimension m_i. Let M_1 be closed so no boundary condition is needed for $\mathfrak{D}_1$. Let $\sigma \in C^\infty(M_2)$. Let*

$$
\begin{aligned}
&M := M_1 \times M_2, && ds^2_M := e^{2\sigma} ds^2_{M_1} + ds^2_{M_2}, \\
&\mathfrak{D} := e^{-2\sigma}\mathfrak{D}_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes \mathfrak{D}_2, && \mathfrak{B} := \mathrm{Id}_1 \otimes \mathfrak{B}_2, \\
&\phi := \phi_1 \otimes \phi_2 \text{ for } \phi_i \in C^\infty(V_i), && \rho := \rho_1 \otimes \rho_2 \text{ for } \rho_i \in C^\infty(V_i^*)\,.
\end{aligned}
$$

If $D_{t,1}\phi_1 = 0$ for all t, then

1. $\beta(\phi, \rho, \mathfrak{D}, \mathfrak{B})(t) = \langle \phi_1, \rho_1\rangle_{L^2} \beta(\phi_2, e^{m_1\sigma}\rho_2, \mathfrak{D}_2, \mathfrak{B}_2)(t)$.

2. $\beta_n(\phi, \rho, \mathfrak{D}, \mathfrak{B}) = \langle \phi_1, \rho_1\rangle_{L^2} \beta_n(\phi_2, e^{m_1\sigma}\rho_2, \mathfrak{D}_2, \mathfrak{B}_2)$.

Proof: It is clear $(\mathfrak{D}, \mathfrak{B})$ is again admissible. Let

$$
u_2 := e^{-t(\mathfrak{D}_2)_{\mathfrak{B}_2}}\phi_2 \quad \text{and} \quad u(x;t) := \phi_1(x_1) \otimes u_2(x_2;t)\,.
$$

We show that $u = e^{-t\mathfrak{D}_{\mathfrak{B}}}\phi$ by computing

$$
\begin{aligned}
&(\partial_t + D_t)u = e^{-2\sigma} D_{t,1}\phi_1 \otimes u_2 + \phi_1 \otimes (\partial_t + D_{t,2})u_2 = 0, \\
&\mathcal{B}_t u = \phi_1 \otimes \mathcal{B}_{t,2} u_2 = 0, \\
&u|_{t=0} = u_1|_{t=0} \otimes \phi_2 = \phi_1 \otimes \phi_2 = \phi\,.
\end{aligned}
$$

The Riemannian measure dx for the warped product metric on M is given by

$$
dx = e^{m_1\sigma} dx_1 dx_2
$$

where dx_i are the Riemannian measures on the factors M_i for $i = 1, 2$. Thus

$$
\begin{aligned}
&\beta(\phi, \rho, \mathfrak{D}, \mathfrak{B})(t) \\
&= \int_M \langle \phi_1(x_1), \rho_1(x_1)\rangle \langle u(x_2;t), \rho_2(x_2)\rangle e^{m_1\sigma} dx_1 dx_2 \\
&= \int_{M_1} \langle \phi_1(x_1), \rho_1(x_1)\rangle dx_1 \cdot \int_{M_2} \langle u(x_2;t), e^{m_1\sigma}\rho_2(x_2)\rangle dx_2 \\
&= \langle \phi_1, \rho_1\rangle_{L^2} \beta(\phi_2, e^{m_1\sigma}\rho_2, \mathfrak{D}_2, \mathfrak{B}_2)(t)\,.
\end{aligned}
$$

The first assertion now follows; the second follows from the first by equating terms in the asymptotic expansions. $\square$

A similar argument can also be used to warp the lower order terms. We introduce some notational conventions. Let $(\theta_1, ..., \theta_{m-1})$ be the usual periodic parameters on the torus $\mathbb{T}^{m-1} := S^1 \times \times S^1$.

Lemma 2.1.10 *Let $M := \mathbb{T}^{m-1} \times M_2$ with $ds_M^2 = g_{uv}(x_2)d\theta^u \circ d\theta^v + ds_{M_2}^2$. Let Δ be the associated scalar Laplacian; the associated connection is the flat connection. Let $\mathcal{B}$ define Dirichlet or Robin boundary conditions. Let $\xi_{u,r}$ be smooth vector fields on M_2. Set $D_t := \Delta + \sum_{r\geq 1}\{G_r^{uu}\partial_u^\theta\partial_v^\theta + \xi_{u,r}\partial_u^\theta + F_r^u\partial_u^\theta\}t^r$. Let $\phi = \phi(x_2)$ and let $\rho = \rho(\theta, x_2)$. Then:*

1. $\beta(\phi, \rho, \mathfrak{D}, \mathcal{B})(t) = \beta(\phi, \rho, \Delta, \mathcal{B})(t)$.
2. $\beta_n(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_n(\phi, \rho, \Delta, \mathcal{B})$.

Proof: Clearly $(\mathfrak{D}, \mathcal{B})$ is admissible on M. We may decompose $\Delta = P_1 + P_2$ where P_2 is a tangential operator on M_2 whose coefficients depend only on x_2 and P_1 is a tangential operator on $\mathbb{T}^{m-1}$ whose coefficients depend on both x_2 and on θ; there are no mixed $\partial_i^x\partial_u^\theta$ terms. Set

$$u_2 := e^{-t(P_{2,\mathcal{B}})}\phi \quad \text{on} \quad M_2 \quad \text{and} \quad u(\theta, x_2; t) := u(x_2; t) \quad \text{on} \quad M\,.$$

We show $u = e^{-t\mathfrak{D}_\mathcal{B}}\phi$ by verifying the defining relations are satisfied

$$\begin{aligned}&(\partial_t + D_t)u = (\partial_t + P_2)u_2 = 0,\\ &\mathcal{B}u = \mathcal{B}u_2 = 0,\\ &u|_{t=0} = u_2|_{t=0} = \phi\,.\end{aligned}$$

It now follows that

$$\beta(\phi, \rho, \mathfrak{D}, \mathcal{B})(t) = \int_M \langle u_2(x_2; t), \rho(x_2, \theta)\rangle dx\,.$$

Setting $G = 0$, $\xi = 0$, and $F = 0$ then yields as a special case that

$$\beta(\phi, \rho, \Delta, \mathcal{B})(t) = \int_M \langle u_2(x_2; t), \rho(x_1, \theta)\rangle dx\,.$$

The Lemma now follows from these two identities. $\square$

We will use the following variant of Lemma 2.1.10 in Section 2.8 in our analysis of oblique boundary conditions.

Lemma 2.1.11 *Let $M := \mathbb{T}^{m-1} \times [0, 1]$ with $ds_M^2 = g_{\alpha\beta}(r)d\theta^\alpha \circ d\theta^\beta + dr^2$. Let Δ be the associated scalar Laplacian. Let $V := M \times \mathbb{C}^\ell$. Let*

$$D := \Delta \otimes \mathrm{Id} + A^\alpha(\theta, r)\partial_\alpha^\theta \quad \textit{and} \quad \mathcal{B} := \partial_r \otimes \mathrm{Id} + B^\alpha(\theta, r)\partial_\alpha^\theta + S$$

where $S \in M_\ell(\mathbb{C})$ is constant and where $A^\alpha, B^\alpha \in C^\infty(\mathrm{End}\,(V))$ are arbitrary. Assume $(D, \mathcal{B})$ is admissible. Let $\phi = \phi(r)$ and $\rho = \rho(\theta, r)$. Then $\beta(\phi, \rho, D, \mathcal{B})(t)$ and $\beta_n(\phi, \rho, D, \mathcal{B})$ are independent of $\{A^\alpha, B^\alpha\}$.

Proof: Decompose $D = P_1 + P_2$ where P_1 is a tangential operator on M_1 and P_2 is a tangential operator on $[0, 1]$ and where $D\phi = P_2\phi$. Set $u_0 := e^{-t(P_2)_{\mathcal{B}_0}}\phi$ where $\mathcal{B}_0 = \partial_r \otimes \mathrm{Id} + S$. The same argument as that given to establish Lemma 2.1.10 then shows $e^{-tD_\mathcal{B}}\phi = u_0$ as well. Thus

$$\beta(\phi, \rho, D, \mathcal{B}) = \int_M \langle u_0(r; t), \rho(\theta, r)\rangle dx\,. \qquad \square$$

2.1.8 Dimensional Analysis

One of the fundamental properties of the heat content coefficients derives from their behavior under rescaling, or, in other words, from dimensional analysis. We use Theorem 1.4.7 to express

$$\beta_n(\phi,\rho,D,\mathcal{B}) = \int_M \beta_n^M(\phi,\rho,D)(x)dx + \int_M \beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y)dy\,.$$

It then follows from the Greiner-Seeley calculus [224, 341] that:

Theorem 2.1.12 *Let $D_c := c^2D$ where c be a positive constant. Then:*
1. $\beta_n^M(\phi,\rho,D_c)(x) = c^n\beta_n^M(\phi,\rho,D)(x)$.
2. $\beta_n^{\partial M}(\phi,\rho,D_c,\mathcal{B})(y) = c^{n-1}\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y)$.

As Theorem 2.1.12 is central to our investigation, it is giving a second proof rather than simply appealing to the Greiner-Seeley calculus. By Lemma 2.1.1, we have, trivially, $\beta_n^M(\phi,\rho,D_c) = c^n\beta_n^M(\phi,\rho,D)$. We can investigate $\beta_n^{\partial M}$ using only the formal properties of this invariant as follows:

Lemma 2.1.13 *Let c be a positive constant. Let $D_c := c^2D$. Then:*
1. $\beta(\phi,\rho,D_c,\mathcal{B})(t) = c^{-m}\beta(\phi,\rho,D,\mathcal{B})(c^2t)$.
2. $\beta_n(\phi,\rho,D_c,\mathcal{B}) = c^{n-m}\beta_n(\phi,\rho,D,\mathcal{B})$.
3. $\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,D_c,\mathcal{B})dy = c^{n-1}\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})dy$.

Proof: The operator D_c is of Laplace type with respect to the Riemannian metric $g_c := c^{-2}g$. Furthermore, $(D_c,\mathcal{B})$ is admissible. Therefore, the heat content asymptotics $\beta_n(\phi,\rho,D_c,\mathcal{B})$ are well-defined.

Let $x = (x_1,...,x_m)$ and let $y = (y_1,...,y_{m-1})$ be local coordinates for M and for the boundary of M, respectively. We let indices μ,ν range from 1 through m and indices α,β range from 1 through $m-1$. The Riemannian measures dx on M and dy on ∂M which are defined by g are then given by

$$dx := \sqrt{\det(g_{\mu\nu})}\cdot dx_1...dx_m \quad\text{and}\quad dy := \sqrt{\det(g_{\alpha\beta})}\cdot dy_1...dy_{m-1}\,.$$

Consequently the Riemannian measures determined by $g_c = c^{-2}g$ are

$$dx_c = c^{-m}dx \quad\text{and}\quad dy_c = c^{1-m}dy\,. \tag{2.1.a}$$

Let $u(x;t) := e^{-tD_\mathcal{B}}\phi$ and let $u_c(x;t) := u(x;c^2t)$. In order to show that $u_c = e^{-tD_{c,\mathcal{B}}}\phi$, we must verify that defining relations are satisfied

$$\begin{aligned}
&(\partial_t + D_c)u_c(x;t) = c^2\partial_2 u(x;c^2t) + c^2Du(x;c^2t) = 0,\\
&\mathcal{B}u_c = \mathcal{B}u = 0,\\
&u_c|_{t=0} = u|_{t=0} = \phi\,.
\end{aligned}$$

We prove Assertion (1) by computing

$$\begin{aligned}
&\beta(\phi,\rho,D_c,\mathcal{B})(t) = \int_M \langle u_c(x;t),\rho(x)\rangle dx_c\\
=\ \ & c^{-m}\int_M\langle u(x;c^2t),\rho(x)\rangle dx = c^{-m}\beta(\phi,\rho,D,\mathcal{B})(c^2t)\,.
\end{aligned}$$

We equate coefficients in the asymptotic expansion to derive Assertion (2) from Assertion (1). By Equation (2.1.a) and Assertion (2),

$$\begin{aligned}
& c^m\beta_n(\phi,\rho,D_c,\mathcal{B}) \\
= & \int_M \beta_n^M(\phi,\rho,D_c)(x)dx + c\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,D_c,\mathcal{B})(y)dy \\
= & c^n\beta_n(\phi,\rho,D,\mathcal{B}) \\
= & c^n\int_M \beta_n^M(\phi,\rho,D)(x)dx + c^n\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y)dy\,.
\end{aligned}$$

Assertion (3) now follows since

$$\beta_n^M(\phi,\rho,D_c)(x) = c^n\beta_n^M(\phi,\rho,D)(x). \qquad \square$$

Remark 2.1.14

1. By replacing $\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y)$ by
$$\tilde\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y) := \lim_{c\to\infty} c^{1-n}\beta_n^{\partial M}(\phi,\rho,D_c,\mathcal{B})(y) \tag{2.1.b}$$
if necessary, and by using Lemma 2.1.13, we may ensure the desired *pointwise* equality of Theorem 2.1.12 (2).
2. In Section 2.2.4, we will use the "weight" to define a grading on the space of invariants; the limit taken in Equation (2.1.b) is just projection on the part of weight $n-1$.
3. If $\mathfrak{D}$ and $\mathfrak{B}$ are a time-dependent family, then we must rescale the parameter to set $\mathfrak{D}_c = \{D_{c^2t}\}$ and $\mathfrak{B}_c = \{\mathcal{B}_{c^2t}\}$. Exactly the same argument then yields
$$\beta(\phi,\rho,\mathfrak{D}_c,\mathfrak{B}_c)(t) = c^{-m}\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(c^2t)\,.$$

2.1.9 Relating Dirichlet and Robin boundary conditions

There is a useful relationship between the heat content invariants for Robin and Dirichlet boundary conditions on the interval.

Lemma 2.1.15 *Let $M := [0,1]$ and let $b \in C^\infty(M)$ be real. Define*

$$\begin{array}{lll}
A := \partial_x + b, & D_1 := A^*A, & \mathcal{B}_1\phi = A\phi|_{\partial M}, \\
A^* := -\partial_x + b, & D_2 := AA^* & \mathcal{B}_2\phi = \phi|_{\partial M}\,.
\end{array}$$

1. $\partial_t\beta(\phi,\rho,D_1,\mathcal{B}_1) = -\beta(A\phi,A\rho,D_2,\mathcal{B}_2)$.

2. $\beta_n(\phi,\rho,D_1,\mathcal{B}_1) = -\frac{2}{n}\beta_{n-2}(A\phi,A\rho,D_2,\mathcal{B}_2)$.

Proof: The operators D_1 and D_2 are formally self-adjoint. The inward unit normal is ∂_x when $x=0$ and $-\partial_x$ when $x=1$. Consequently, the boundary condition defined by the operator $\mathcal{B}_1$ is the Robin boundary condition where

$$S(0) = b(0) \quad \text{and} \quad S(1) = -b(1)\,.$$

The operator $\mathcal{B}_2$ defines Dirichlet boundary conditions. Thus the discussion in Sections 1.5.1 and 1.5.2 shows that D_1 and D_2 are self-adjoint, non-negative operators and that $(D_1,\mathcal{B}_1)$ and $(D_2,\mathcal{B}_2)$ are admissible.

Let $\{\phi_\nu, \lambda_\nu\}$ be the discrete spectral resolution of $(D_1, \mathcal{B}_1)$ as described in Theorem 1.4.18. We restrict to $\lambda_\nu > 0$ as the zero spectrum plays no role. Let

$$\psi_\nu := \tfrac{1}{\sqrt{\lambda_\nu}} A\phi_\nu .$$

Then $\{\psi_\nu, \lambda_\nu\}$ comprises a discrete spectral resolution of $(D_2, \mathcal{B}_2)$ on the space $\ker(D_2)^\perp :=$ Range (A). Let $\sigma_{1,\nu}$ and $\sigma_{2,\nu}$ be the associated Fourier coefficients of $(D_1, \mathcal{B}_1)$ and $(D_2, \mathcal{B}_2)$, respectively. Let $f \in C^\infty(M)$. Since $A\phi_\nu|_{\partial M} = 0$, we may integrate by parts to see

$$\begin{aligned}
\sigma_{2,\nu}(Af) &= \int_M (Af, \psi_\nu)dx = \frac{1}{\sqrt{\lambda_\nu}} \int_M (Af, A\phi_\nu)dx \\
&= \frac{1}{\sqrt{\lambda_\nu}} \int_M (f, A^* A\phi_\nu)dx = \sqrt{\lambda_\nu} \int_M (f, \phi_\nu) \\
&= \sqrt{\lambda_\nu} \cdot \sigma_{1,\nu}(f) .
\end{aligned}$$

Assertion (1) now follows from Lemma 2.1.6 since

$$\begin{aligned}
&\partial_t \beta(\phi, \rho, D_1, \mathcal{B}_1) = \partial_t \sum_\nu e^{-t\lambda_\nu} \sigma_{1,\nu}(\phi)\sigma_{1,\nu}(\rho) \\
= &-\sum_\nu e^{-t\lambda_\nu} \lambda_\nu \sigma_{1,\nu}(\phi)\sigma_{1,\nu}(\rho) = -\sum_\nu e^{-t\lambda_\nu} \sigma_{2,\nu}(A\phi)\sigma_{2,\nu}(A\rho) \\
= &-\beta(A\phi, A\rho, D_2, \mathcal{B}_2)(t) .
\end{aligned}$$

Assertion (2) follows from Assertion (1) by equating coefficients in the associated asymptotic expansions. □

2.1.10 Shuffle formulae for non-minimal operators

As discussed in Section 1.6.7, let

$$D := Ad\delta + B\delta d$$

where A and B are positive constants. This is a *non-minimal* operator on $C^\infty(\Lambda(M))$; if $A \neq B$, then D is not of Laplace type but D is still elliptic with respect to the cone $\mathcal{C}$.

Lemma 2.1.16 *Let $D := Ad\delta + B\delta d$, where A and B are positive constants, and let $\Delta = d\delta + \delta d$ on $C^\infty(\Lambda(M))$. Let $\phi, \rho \in C^\infty(\Lambda(M))$.*

1. *If $\mathcal{B}$ defines absolute boundary conditions, then*

$$\beta(d\phi, \rho, D, \mathcal{B})(t) = \beta(d\phi, \rho, \Delta, \mathcal{B})(At) \text{ and}$$
$$\beta_n(d\phi, \rho, D, \mathcal{B}) = A^{n/2}\beta_n(d\phi, \rho, \Delta, \mathcal{B}).$$

2. *If $\mathcal{B}$ defines relative boundary conditions, then*

$$\beta(\delta\phi, \rho, D, \mathcal{B})(t) = \beta(\delta\phi, \rho, \Delta, \mathcal{B})(At) \text{ and}$$
$$\beta_n(\delta\phi, \rho, D, \mathcal{B}) = B^{n/2}\beta_n(\delta\phi, \rho, \Delta, \mathcal{B}).$$

Proof: Let $\mathcal{B}$ define absolute boundary conditions. By Theorem 1.5.7, there is a complete orthonormal basis $\{\rho_i^+ \rho_j^-, \rho_k^0\}$ for $L^2(\Lambda(M))$ so that

$$\begin{array}{lll} d\delta\rho_i^+ = \lambda_i^+\rho_i^+, & d\rho_i^+ = 0, & \mathcal{B}\rho_i^+ = 0, \\ \delta\rho_j^- = 0, & \delta d\rho_j^- = \lambda_j^-\rho_j^-, & \mathcal{B}\rho_j^- = 0, \\ \delta\rho_k^0 = 0, & d\rho_k^0 = 0, & \mathcal{B}\rho_k^0 = 0\,. \end{array}$$

Let $\sigma_i^+ := (d\phi, \rho_i^+)_{L^2}$ be the Fourier coefficients. We have $\delta\rho_j^- = 0$ and $\delta\rho_k^0 = 0$. Since $\mathcal{B}\rho_j^- = 0$ and $\mathcal{B}\rho_k^0 = 0$, $\mathfrak{i}(e_m)\rho_j^- = 0$ and $\mathfrak{i}(e_m)\rho_k^0 = 0$ on ∂M. Thus, by Lemma 1.4.16, the remaining Fourier coefficients of $d\phi$ vanish as

$$\int_M (d\phi, \rho_j^-)dx = \int_M (\phi, \delta\rho_j^-)dx - \int_{\partial M} (\phi, \mathfrak{i}(e_m)\rho_j^-)dy = 0,$$
$$\int_M (d\phi, \rho_k^0)dx = \int_M (\phi, \delta\rho_k^0)dx - \int_{\partial M} (\phi, \mathfrak{i}(e_m)\rho_k^0)dy = 0\,.$$

Consequently,

$$d\phi = \sum_i \sigma_i^+ \rho_i^+$$

in the distributional sense. Let

$$u(x;t) := \sum_i e^{-t\lambda_i^+} \sigma_i^+ \rho_i^+\,.$$

Since $\Delta\rho_i^+ = \lambda_i^+\rho_i^+$, $D\rho_i^+ = A\lambda_i^+\rho_i^+$, and $\mathcal{B}\rho_i^+ = 0$, one may verify

$$u(x;t) = e^{-t\Delta_{\mathcal{B}}} d\phi \quad \text{and} \quad u(x;At) = e^{-tD_{\mathcal{B}}} d\phi$$

by computing that

$$(\partial_t + d\delta + \delta d)u(x;t) = \sum_i e^{-t\lambda_i^+} \sigma_i^+ (-\lambda_i^+ + \lambda_i^+)\rho_i^+ = 0,$$

$$(\partial_t + Ad\delta + B\delta d)u(x;At) = \sum_i e^{-At\lambda_i^+} \sigma_i^+ (-A\lambda_i^+ + A\lambda_i^+)\rho_i^+ = 0,$$

$$\mathcal{B}u = \sum_i e^{-t\lambda_i^+} \sigma_i^+ \mathcal{B}\rho_i^+ = 0,$$

$$u|_{t=0} = \sum_i \sigma_i^+ \rho_i^+ = d\phi\,.$$

The first equality of Assertion (1) now follows since

$$\begin{aligned} \beta(d\phi, \rho, D, \mathcal{B})(t) &= \int_M (u(x;At), \rho)dx \\ &= \beta(d\phi, \rho, \Delta, \mathcal{B})(At)\,. \end{aligned}$$

The second equality of Assertion (1) follows by equating terms in the asymptotic expansions. Assertion (2) follows from Assertion (1) by duality, since the Hodge operator interchanges relative and absolute boundary conditions. □

2.2 Functorial properties II

We continue our discussion of functorial properties of the heat content asymptotics in this section.

2.2.1 Transmission boundary conditions

We recall the notational conventions from Section 1.6.1. Let $M = (M_+, M_-)$, $V = (V_+, V_-)$, and $D = (D_+, D_-)$ where $D_\pm$ are operators of Laplace type on bundles $V_\pm$ over $M_\pm$. We assume as compatibility conditions that

$$\partial M_+ = \partial M_- = \Sigma, \quad g_+|_\Sigma = g_-|_\Sigma, \quad \text{and} \quad V_+|_\Sigma = V_-|_\Sigma\,. \tag{2.2.a}$$

Let $\nu_\pm$ be the inward unit normals of Σ in $\nu_\pm$; $\nu_+ + \nu_- = 0$. If U is an impedance matching endomorphism on Σ, then the *transmission boundary operator* $\mathcal{B}_U$ is defined for $\phi = (\phi_+, \phi_-)$ by setting

$$\mathcal{B}_U\phi := \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \ \oplus\ \{\nabla_{\nu_+}\phi_+|_\Sigma + \nabla_{\nu_-}\phi_-|_\Sigma - U\phi_+|_\Sigma\}\,.$$

We double a model problem (M_0, V_0, D_0) to obtain our first functorial property by considering the singular structures. Over M_0, we let

$$\mathcal{B}_D\phi_0 := \phi_0|_{\partial M_0} \quad \text{and} \quad \mathcal{B}_R\phi_0 := (\nabla_{e_m} + S)\phi_0|_{\partial M_0}$$

define Dirichlet and Robin boundary conditions, respectively. Let

$$\phi_{\text{even}}(x) := \tfrac{1}{2}(\phi_+(x_+) + \phi_-(x_-)) \quad \text{and} \quad \phi_{\text{odd}}(x) = \tfrac{1}{2}(\phi_+(x_+) - \phi_-(x_-))$$

be the even and odd parts of $\phi = (\phi_+, \phi_-)$. The following Lemma relates the heat content asymptotics on M to those on M_0 taking into account the $\mathbb{Z}_2$ action which interchanges x_+ and x_-.

Lemma 2.2.1 *Adopt the notation given above. If $U = -2S$, then:*

1. $\beta(\phi, \rho, D, \mathcal{B}_U)(t) = 2\beta(\phi_{\text{odd}}, \rho_{\text{odd}}, D, \mathcal{B}_D)(t) + 2\beta(\phi_{\text{even}}, \rho_{\text{even}}, D, \mathcal{B}_R)(t)$.
2. $\beta_n(\phi, \rho, D, \mathcal{B}_U) = 2\beta(\phi_{\text{odd}}, \rho_{\text{odd}}, D, \mathcal{B}_D) + 2\beta(\phi_{\text{even}}, \rho_{\text{even}}, D, \mathcal{B}_R)$.

Proof: Let $u = e^{-tD_{\mathcal{B}_U}}\phi$ solve the heat equation on the singular manifold M with transmission boundary conditions. Decompose u into even and odd pieces by setting

$$\begin{aligned} u_{\text{even}}(x;t) &:= \tfrac{1}{2}\{u_+(x_+;t) + u_-(x_-;t)\} \quad \text{and} \\ u_{\text{odd}}(x;t) &:= \tfrac{1}{2}\{u_+(x_+;t) - u_-(x_-;t)\}\,. \end{aligned}$$

We wish to show that

$$u_{\text{even}} = e^{-tD_{0,\mathcal{B}_R}}\phi_{\text{even}} \qquad \text{and} \qquad u_{\text{odd}} = e^{-tD_{0,\mathcal{B}_D}}\phi_{\text{odd}}\,. \tag{2.2.b}$$

We note that $u_+|_\Sigma = u_-|_\Sigma$. We compute that

$$(\partial_t + D_0)u_{\text{even}} = \tfrac{1}{2}\{(\partial_t + D_+)u_+(x_+;t) + (\partial_t + D_-)u_-(x_-;t)\} = 0,$$

$$
\begin{aligned}
&(\nabla_\nu + S)u_{\text{even}}|_\Sigma = \tfrac{1}{2}\{(\nabla_{\nu_+} + S)u_+|_\Sigma + (\nabla_{\nu_-} + S)u_-|_\Sigma\} \\
&\qquad = \tfrac{1}{2}\{\nabla_{\nu_+}u_+ + \nabla_{\nu_-}u_- - Uu_-\}|_\Sigma = 0, \\
&u_{\text{even}}|_{t=0} = \phi_{\text{even}}\,,
\end{aligned}
$$

and that

$$
\begin{aligned}
&(\partial_t + D_0)u_{\text{odd}} = \tfrac{1}{2}\{(\partial_t + D_+)u_+(x_+;t) - (\partial_t + D_-)u_-(x_-;t)\} = 0, \\
&u_{\text{odd}}|_\Sigma = \tfrac{1}{2}(u_+|_\Sigma - u_-|_\Sigma) = 0, \\
&u_{\text{odd}}|_{t=0} = \phi_{\text{odd}}\,.
\end{aligned}
$$

We then have

$$
\begin{aligned}
& \beta(\phi,\rho,D,\mathcal{B}_U)(t) \\
= & \int_{M_+}\langle u_+(x_+;t),\rho_+(x_+)\rangle dx_+ + \int_{M_-}\langle u_-(x_-;t),\rho_-(x_-)\rangle dx_- \\
= & \tfrac{1}{2}\int_{M_0}\langle u_+(x_+;t) + u_-(x_-;t),\rho_+(x_+) + \rho_-(x_-)\rangle dx_0 \\
+ & \tfrac{1}{2}\int_{M_0}\langle u_+(x_+;t) - u_-(x_-;t),\rho_+(x_+) - \rho_-(x_-)\rangle dx_0 \\
= & 2\int_{M_0}\left\{\langle u_{\text{even}}\,,\rho_{\text{even}}\,\rangle + \langle u_{\text{odd}}\,,\rho_{\text{odd}}\,\rangle\right\}dx \\
= & 2\beta(\phi_{\text{even}}\,,\rho_{\text{even}}\,,D_0,\mathcal{B}_R)(t) + 2\beta(\phi_{\text{odd}}\,,\rho_{\text{odd}}\,,D_0,\mathcal{B}_D)(t)\,.
\end{aligned}
$$

This proves the first assertion; the second now follows. □

The null space of the operators in question again plays a crucial role.

Lemma 2.2.2 *Let $D_\pm$ be the scalar Laplacians on the manifolds $M_\pm$. Let $\phi = 1$ and let $U = 0$. Then:*

1. $\beta(\phi,\rho,D,\mathcal{B}_U)(t) = \text{vol}\,(M)$.
2. $\beta_0(\phi,\rho,D,\mathcal{B}_U) = \text{vol}\,(M)$.
3. *If $n \geq 1$, then $\beta_n(\phi,\rho,D,\mathcal{B}_U) = 0$.*

Proof: Let $\phi(x) = u(x;t) = 1$. We verify $u = e^{-tD_{\mathcal{B}_U}}\phi$ by checking the defining relations are satisfied:

$$
\begin{aligned}
&(\partial_t + D)u = 0, \\
&u_+|_\Sigma - u_-|_\Sigma = 0, \\
&u_{+;\nu_+}|_\Sigma + u_{-;\nu_-}|_\Sigma = 0 + 0 = 0, \\
&u|_{t=0} = \phi\,.
\end{aligned}
$$

The first assertion of the Lemma is now immediate; the second follows from the first. □

If we introduce an "artificial" singularity, then the heat content asymptotics are not affected.

Lemma 2.2.3 *Let D_0 be an operator of Laplace type on a bundle V_0 over a closed Riemannian manifold M_0. Let Σ be a smooth submanifold of M_0 which separates M_0 into two components M_+ and M_-. Let $D_\pm := D_0|_{M_\pm}$ and let $U = 0$. Let $\phi_0 \in C^\infty(V_0)$ and $\rho_0 \in C^\infty(V_0^*)$. Set $\phi_\pm := \phi_0|_{M_\pm}$ and $\rho_\pm := \rho_0|_{M_\pm}$. Then*

1. $\beta(\phi, \rho, D, \mathcal{B}_U)(t) = \beta(\phi_0, \rho_0, D_0)(t)$.
2. $\beta_n(\phi, \rho, D, \mathcal{B}_U) = \beta_n(\phi_0, \rho_0, D_0)$.

Proof: Let $u_0 := e^{-tD_0}\phi_0$. Let $u_\pm := u_0|_{M_\pm}$. We check $u = e^{-tD_{\mathcal{B}_U}}\phi$ by verifying that the defining relations hold

$$\begin{aligned}
&(\partial_t + D_\pm)u_\pm = (\partial_t + D_0)u_0|_{M_\pm} = 0,\\
&u_+|_\Sigma - u_-|_\Sigma = u_0|_\Sigma - u_0|_\Sigma = 0,\\
&u_{+;\nu_+}|_\Sigma + u_{-;\nu_-}|_\Sigma = u_{0;\nu_+}|_\Sigma - u_{0;\nu_+}|_\Sigma = 0,\\
&u_\pm|_{t=0} = u_0|_{t=0,M_\pm} = \phi_0|_{M_\pm} = \phi_\pm\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\beta(\phi, \rho, D, \mathcal{B}_U)(t)\\
= &\int_{M_+} \langle u_+(x_+; t), \rho_+(x_+)\rangle dx_+ + \int_{M_-} \langle u_-(x_-; t), \rho_-(x_-)\rangle dx_-\\
= &\int_{M_0} \langle u_0(x; t), \rho_0(x)\rangle dx_0 = \beta(\phi_0, \rho_0, D_0)(t)\,.
\end{aligned}$$

This establishes Assertion (1); Assertion (2) follows from Assertion (1). □

2.2.2 Transfer boundary conditions

We adopt the same notation as that used to discuss transmission boundary conditions with the exception that we replace the compatibility condition of Equation (2.2.a) by the weaker assumption that

$$\partial M_+ = \partial M_- = \Sigma \quad \text{and} \quad g_+|_\Sigma = g_-|_\Sigma\,.$$

No relation is assumed between $V_+|_\Sigma$ and $V_-|_\Sigma$. We adopt the notation of Section 1.6.3 and define

$$\mathcal{B}_\mathcal{S}\phi := \left\{ \begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \right\}\Bigg|_\Sigma$$

where

$$\begin{array}{ll}
S_{++} : V_+|_\Sigma \to V_+|_\Sigma, & S_{+-} : V_-|_\Sigma \to V_+|_\Sigma,\\
S_{-+} : V_+|_\Sigma \to V_-|_\Sigma, & S_{--} : V_-|_\Sigma \to V_-|_\Sigma.
\end{array}$$

Lemma 2.2.2 generalizes to this setting.

Lemma 2.2.4 *Let $D_\pm$ be the scalar Laplacians on the manifolds $M_\pm$. Let $\phi = 1$ and let $\rho = 1$. If $S_{++} + S_{+-} = 0$ and if $S_{--} + S_{-+} = 0$, then:*

1. $\beta(\phi, \rho, D, \mathcal{B}_\mathcal{S})(t) = \mathrm{vol}\,(M)$.

2. $\beta_0(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = \mathrm{vol}\,(M)$.

3. *If* $n \geq 1$, *then* $\beta_n(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = 0$.

Proof: Let $u(x;t) = 1$. We verify $u = e^{-tD_\mathcal{B}}\phi$ by checking the defining relations are satisfied:

$$\begin{aligned}
&(\partial_t + D)u = 0,\\
&\phi_{+;\nu_+}|_\Sigma + S_{++}\phi_+|_\Sigma + S_{+-}\phi_-|_\Sigma = 0,\\
&\phi_{-;\nu_-}|_\Sigma + S_{--}\phi_-|_\Sigma + S_{-+}\phi_+|_\Sigma = 0,\\
&u|_{t=0} = \phi\,.
\end{aligned}$$

The Lemma now follows. □

Let $\mathcal{B}_{R(S_{++})}$ and $\mathcal{B}_{R(S_{--})}$ define Robin boundary conditions for $D_\pm$. If $S_{+-} = 0$ and $S_{-+} = 0$, then $\mathcal{B}_\mathcal{S} = \mathcal{B}_{R(S_{++})} \oplus \mathcal{B}_{R(S_{--})}$. Since the operators and boundary conditions decouple, the following Lemma is immediate:

Lemma 2.2.5 *If* $S_{+-} = 0$ *and if* $S_{-+} = 0$, *then*

1. $\beta(\phi,\rho,D,\mathcal{B}_\mathcal{S})(t) = \beta(\phi_+,\rho_+,D_+,\mathcal{B}_{R(S_{++})})(t) + \beta(\phi_-,\rho_-,D_-,\mathcal{B}_{R(S_{--})})(t)$.

2. $\beta_n(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = \beta_n(\phi_+,\rho_+,D_+,\mathcal{B}_{R(S_{++})}) + \beta_n(\phi_-,\rho_-,D_-,\mathcal{B}_{R(S_{--})})$.

2.2.3 Time-dependent processes

We now discuss functorial properties that relate directly to time-dependent as opposed to static processes. The first property deals with a rather trivial variation of the geometry.

Lemma 2.2.6 *Let* $(D,\mathcal{B})$ *be elliptic with respect to the cone* $\mathcal{C}_\delta$. *Consider the family* $D_t := e^t D_0$ *and* $\mathcal{B}_t := \mathcal{B}_0$. *Then:*

1. $\beta(\phi,\rho,\mathfrak{D},\mathfrak{B})(t) = \beta(\phi,\rho,D,\mathcal{B})(e^t - 1)$.

2. $\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_n(\phi,\rho,D,\mathcal{B})$ *for* $n \leq 2$.

3. $\beta_3(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_3(\phi,\rho,D,\mathcal{B}) + \frac{1}{4}\beta_1(\phi,\rho,D,\mathcal{B})$.

4. $\beta_4(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_4(\phi,\rho,D,\mathcal{B}) + \frac{1}{2}\beta_2(\phi,\rho,D,\mathcal{B})$.

Proof: Let $s(t) := e^t - 1$. Then $\partial_t = e^t\partial_s$ and $s(0) = 0$. Let $u(x;t) := e^{-tD_\mathcal{B}}\phi$ and let $v(x;t) := u(x;s(t))$. We show $v = e^{-t\mathfrak{D}_\mathfrak{B}}\phi$ by verifying that the defining relations are satisfied

$$\begin{aligned}
&(\partial_t + D_t)v(x;t) = e^t\{(\partial_s + D_0)u(x;s)\} = 0,\\
&\mathcal{B}_t v(y;t) = \mathcal{B}_0 u(y;s(t)) = 0,\\
&v|_{t=0} = u|_{t=0} = \phi\,.
\end{aligned}$$

The first assertion now follows. We expand

$$\begin{array}{ll}
s = t + \frac{1}{2}t^2 + O(t^3), & s^2 = t^2 + O(t^3)\\
s^{1/2} = t^{1/2} + \frac{1}{4}t^{3/2} + O(t^{5/2}), & s^{3/2} = t^{3/2} + O(t^{5/2})\,.
\end{array}$$

We equate coefficients of t in the asymptotic expansions

$$\sum_{n=0}^{\infty} \beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B})t^{n/2} \sim \sum_{n=0}^{\infty} \beta_n(\phi,\rho,D,\mathcal{B})s^{n/2}$$

to derive the remaining assertions. □

We now make a time-dependent gauge transformation.

Lemma 2.2.7 *Let $(\mathfrak{D},\mathfrak{B})$ be admissible on M. Let $f \in C^\infty(M)$. Define $\check{\mathfrak{D}}$ by setting $\check{D}_t := e^{tf}(\partial_t + D_t)e^{-tf} - \partial_t$ and $\check{\mathfrak{B}}$ by setting $\check{\mathcal{B}}_t := e^{tf}\mathcal{B}_t e^{-tf}$. Then*

1. $\beta_n(\phi,\rho,\check{\mathfrak{D}},\check{\mathfrak{B}}) = \beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B})$ *for* $n \le 2$.
2. $\beta_3(\phi,\rho,\check{\mathfrak{D}},\check{\mathfrak{B}}) = \beta_3(\phi,\rho,\mathfrak{D},\mathfrak{B}) + \beta_1(\phi,f\rho,\mathfrak{D},\mathfrak{B})$.
3. $\beta_4(\phi,\rho,\check{\mathfrak{D}},\check{\mathfrak{B}}) = \beta_4(\phi,\rho,\mathfrak{D},\mathfrak{B}) + \beta_2(\phi,f\rho,\mathfrak{D},\mathfrak{B}) + \frac{1}{2}\beta_0(\phi,f^2\rho,\mathfrak{D},\mathfrak{B})$.

Proof: Let $u(x;t) := e^{-t\mathfrak{D}_{\mathfrak{B}}}\phi$. Set $\check{u}(x;t) := e^{tf}u(x;t)$. We verify $\check{u} = e^{-t\check{\mathfrak{D}}_{\check{\mathfrak{B}}}}$ by checking the defining equations are satisfied

$$\begin{aligned}
&(\partial_t + \check{D}_t)\check{u} = e^{tf}(\partial_t + D_t)e^{-tf}e^{tf}u(x;t) = 0,\\
&\check{\mathcal{B}}_t\check{u} = e^{tf}\mathcal{B}_t e^{-tf}e^{tf}u = 0,\\
&\check{u}|_{t=0} = u|_{t=0} = \phi\,.
\end{aligned}$$

Consequently, we may compute:

$$\begin{aligned}
\beta(\phi,\rho,\check{\mathfrak{D}},\check{\mathfrak{B}})(t) &= \int_M \langle e^{tf}u(x;t),\rho(x)\rangle dx\\
&\sim \sum_{r=0}^{\infty}\frac{1}{r!}\int_M \langle u(x;t), f^r\rho\rangle dx \cdot t^r\\
&\sim \sum_{r=0}^{\infty}\frac{1}{r!}\beta(\phi,f^r\rho,\mathfrak{D},\mathfrak{B})(t)\cdot t^r\,.
\end{aligned}$$

The assertions of the Lemma now follow by equating powers of t in the asymptotic expansions. □

We make a gauge transformation in the spacial coordinates and warp the higher order terms.

Lemma 2.2.8 *Let $M := S^1 \times [0,1]$. Let ε be an auxiliary real parameter. Set*

$$D_t := -\partial_r^2 - \partial_\theta^2 - 1 + \varepsilon t\partial_r(\partial_\theta - \sqrt{-1})\,.$$

Let $\mathcal{B}$ define Dirichlet or Robin boundary conditions (with $S = S(r)$), let $\phi(r,\theta) := e^{\sqrt{-1}\theta}\phi_0(r)$, and let $\rho(r,\theta) := e^{-\sqrt{-1}\theta}\rho_0(r)\rho_1(\theta)$. Then:

1. $\beta(\phi,\rho,\mathfrak{D},\mathcal{B})(t) = \beta(\phi_0,\rho_0,-\partial_r^2,\mathcal{B})(t)\cdot\int_{S^1}\rho_1(\theta)d\theta$.
2. $\beta_n(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_n(\phi_0,\rho_0,-\partial_r^2,\mathcal{B})\cdot\int_{S^1}\rho_1(\theta)d\theta$.

Proof: Let $u_0 := e^{-t(-\partial_r^2)_{\mathcal{B}}}\phi_0$ on $[0,1]$ and let $u(r,\theta;t) := e^{\sqrt{-1}\theta}u_0(r;t)$. We show that $u = e^{-t\mathfrak{D}_{\mathcal{B}}}\phi$ by computing:

$$\begin{aligned}
(\partial_t + D_t)u &= \{\partial_t - \partial_r^2 - \partial_\theta^2 - 1 + \varepsilon t\partial_r(\partial_\theta - \sqrt{-1})\}e^{\sqrt{-1}\theta}u_0(r;t)\\
&= e^{\sqrt{-1}\theta}\{\partial_t - \partial_r^2\}u_0(r;t) = 0,\\
\mathcal{B}u &= e^{\sqrt{-1}\theta}\mathcal{B}u_0 = 0,\\
u|_{t=0} &= e^{\sqrt{-1}\theta}\phi_0 = \phi\,.
\end{aligned}$$

We prove the first assertion by computing:

$$\begin{aligned}
\beta(\phi,\rho,\mathfrak{D},\mathcal{B})(t) &= \int_M \langle u(r,\theta;t),\rho\rangle dr d\theta\\
&= \int_M \langle u_0(r;t),\rho_0(r)\rangle dr \cdot \int_{S^1}\rho_1(\theta)d\theta\\
&= \beta(\phi_0,\rho_0,-\partial_r^2,\mathcal{B})\cdot\int_{S^1}\rho_1(\theta)d\theta\,.
\end{aligned}$$

The second assertion now follows from the first by equating coefficients of t in the asymptotic expansions. □

The following Lemma involves making a change of variables which mixes up the temporal and spatial coordinates. Let x be the usual coordinate on $M = [0,1]$. Let F be a non-negative smooth function on M so that

$$F(0) = F(1) = 0 \quad \text{and} \quad F \equiv 0 \quad \text{near} \quad x = 1\,.$$

We introduce new coordinates $(\bar{x},\bar{t})$ on $[0,1]\times[0,\varepsilon)$ by setting

$$(\bar{x},\bar{t}) := (x + tF(x), t)\,.$$

The Jacobian J of this coordinate transformation is then given by

$$J := \begin{pmatrix} \partial_x\bar{x} & \partial_t\bar{x}\\ \partial_x\bar{t} & \partial_t\bar{t}\end{pmatrix} = \begin{pmatrix} 1+tF_x & F\\ 0 & 1\end{pmatrix}.$$

If t is small, then $\det(J) \neq 0$ so this is an admissible change of variables. We have

$$\begin{aligned}
&d\bar{x} = (1+tF_x)dx + Fdt, && d\bar{t} = dt,\\
&\partial_{\bar{x}} = (1+tF_x)^{-1}\partial_x, && \partial_{\bar{t}} = \partial_t - F(1+tF_x)^{-1}\partial_x\,.
\end{aligned} \tag{2.2.c}$$

Lemma 2.2.9 *Let $M = [0,1]$. Let $\Delta := -\partial_{\bar{x}}^2$. Let*

$$D_t := -\{(1+tF_x)^{-1}\partial_x\}^2 - F(1+tF_x)^{-1}\partial_x\,.$$

Let $\mathcal{B}$ define Dirichlet boundary conditions, let $\phi = 1$, and let $\rho = 1$. Then:

1. $\beta(\phi,\rho,\Delta,\mathcal{B})(t) = \beta(\phi,\rho,\mathfrak{D},\mathcal{B})(t) + t\beta(1,F_x,\mathfrak{D},\mathcal{B})(t)$.

2. $\beta_n(\phi,\rho,\Delta,\mathcal{B}) = \beta_n(\phi,\rho,\mathfrak{D},\mathcal{B}) + \beta_{n-2}(1,F_x,\mathfrak{D},\mathcal{B})$.

Proof: We let $u = e^{-t\Delta_{\mathcal{B}}}1$ and define $v(x;t) = u(x+tF(x),t)$. We show $v = e^{-t\mathfrak{D}_{\mathcal{B}}}\phi$ by changing variables and using Display (2.2.c) to see

$$(\partial_t + D_t)v(x;t) = (\partial_{\bar{t}} + \Delta)u(\bar{x},\bar{t}) = 0,$$

$$\mathcal{B}v(x;t) = v|_{\partial M} = u|_{\partial M} = 0,$$
$$v|_{t=0} = u|_{t=0} = 1\,.$$

Fix t. It now follows that

$$\begin{aligned}\beta(\phi,\rho,\Delta,\mathcal{B})(t) &= \int_M u(\bar{x},t)d\bar{x} = \int_M u(x+tF(x);t)(1+tF_x)dx\\ &= \int_M v(x;t)dx + t\int_M v(x;t)F_x dx\\ &= \beta(\phi,\rho,\mathfrak{D},\mathcal{B})(t) + t\beta(1,F_x,\mathfrak{D},\mathcal{B})(t)\,.\end{aligned}$$

The first assertion now follows; the second follows from the first by equating terms in the asymptotic expansions. □

We now exploit a crucial fact concerning Neumann boundary conditions.

Lemma 2.2.10 *Let $M = S^1 \times [0,1]$. Let $\mathcal{B}_0 = \partial_r$ and let*

$$D_t := -\partial_r^2 - \partial_\theta^2 - 1 + t(\mathcal{G}_{22}\partial_r^2 + 2\mathcal{G}_{1,12}\partial_r\partial_\theta + \mathcal{F}_{1,2}\partial_r)\,.$$

Let $\phi(r,\theta) = e^{\sqrt{-1}\theta}$. Then $\beta_n(\phi,\rho,\mathfrak{D},\mathcal{B}) = 0$ for $n > 0$.

Proof: Let $u(r,\theta;t) := e^{\sqrt{-1}\theta}$. It is immediate that u satisfies the defining relations so $u = e^{-t\mathfrak{D}_{\mathcal{B}}}\phi$. Thus β is independent of the parameter t. □

A similar proof also yields:

Lemma 2.2.11 *Let*

$$\begin{aligned}&D_t u := \Delta u + \textstyle\sum_{r\geq 1} t^r(\mathcal{G}_{r,ij}u_{;ij} + \mathcal{F}_{r,i}u_{;i}\}, \quad \textit{and}\\ &\mathcal{B}_t u := \{u_{;m} + \textstyle\sum_{r\geq 1} t^r\Gamma_{a,r}u_{;a}\}|_{\partial M}\,.\end{aligned}$$

Let $\phi = 1$. Then $\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B})$ is independent of $\{\mathcal{G},\ \mathcal{F},\ \Gamma\}$.

2.2.4 Expressing the invariants $\beta_n^{\partial M}$ relative to a Weyl basis

For the remainder of this section, we restrict our attention to operators of Laplace type and to the boundary conditions discussed in Sections 1.5 and 1.6. We first deal with Dirichlet boundary conditions as there are no additional structures present. Let $\mathcal{B}\phi := \phi|_{\partial M}$ be the Dirichlet boundary operator. Let $\{e_1,...,e_m\}$ be a local orthonormal frame for TM which is normalized so e_m is the inward unit normal vector field on ∂M. We recall our previous notational conventions. Let indices i,j,k range from 1 through m and index this frame field. We let indices a,b,c range from 1 to $m-1$ and index the induced orthonormal frame for the tangent bundle of the boundary. Let ";" denote the components of multiple covariant differentiation of tensors of all types with respect to the Levi-Civita connection defined by M and the connection defined by D. Similarly, we use the Levi-Civita connection of ∂M and the connection defined by D to tangentially covariantly differentiate tensors defined on ∂M; denote the components of multiple covariant differentiation in this case by ":".

If there are no indices present from the tangent bundle, then ":" and ";" agree. Thus, for example, $\phi_{:a} = \phi_{;a}$. More generally, the difference between ":" and ";" is measured by the second fundamental form. By Lemma 1.1.4,

$$\phi_{;ab} = \phi_{:ab} - L_{ab}\phi_{;m} \,.$$

Because the second fundamental form L is defined only on ∂M, $L_{ab;c}$ is not defined and we use instead the tensor $L_{ab:c}$.

Let R_{ijkl} be the components of the curvature tensor of the Levi-Civita connection, let Ω_{ij} be the components of the curvature operator of the connection defined by D, let E be the endomorphism defined by D, and let L_{ab} be the components of the second fundamental form. We may covariantly differentiate $\{\phi, \rho, R, \Omega, E\}$ in arbitrary directions. On the other hand, the second fundamental form L is defined only on ∂M. Thus, as noted above, we may only covariantly differentiate L tangentially with respect to the Levi-Civita connection of ∂M.

An appropriate generalization of Theorem 1.1.1 then permits us to see that any invariant of the partial derivatives of the structures involved can be expressed in terms of the variables

$$\{\phi_{;j_1\dots j_\ell},\ \rho_{;j_1\dots j_\ell},\ R_{i_1 i_2 i_3 i_4;j_1\dots j_\ell},\ \ \Omega_{;j_1\dots j_\ell},\ E_{;j_1\dots j_\ell},\ L_{a_1 a_2:b_1\dots b_\ell}\} \tag{2.2.d}$$

relative to a suitable coordinate system and suitable frame for V. Consequently, $\beta_n^{\partial M}(\phi, \rho, D, \mathcal{B})$ is an invariant polynomial in these variables. The analysis performed in Section 1.7.2 is relevant. The Levi-Civita connections of the metric g and $g_c := c^{-2}g$ are the same. The connections determined by D and $D_c := c^2 D$ are the same. Let $e_c = (ce_1, ..., ce_m)$. Then

$$\begin{aligned}
&\phi_{;j_1\dots j_\ell}(g_c, e_c) = c^\ell \phi_{;j_1\dots j_\ell}(g, e),\\
&\rho_{;j_1\dots j_\ell}(g_c, e_c) = c^\ell \rho_{;j_1\dots j_\ell}(g, e),\\
&R_{i_1 i_2 i_3 i_4;j_1\dots j_\ell}(g_c, e_c) = c^{2+\ell} R_{i_1 i_2 i_3 i_4;j_1\dots j_\ell}(g, e),\\
&L_{a_1 a_2:b_1\dots b_\ell}(g_c, e_c) = c^{1+\ell} L_{a_1 a_2:b_1\dots b_\ell}(g, e),\\
&\Omega_{i_1 i_2;j_1\dots j_\ell}(g_c, e_c, D_c) = c^{2+\ell}\Omega_{i_1 i_2;j_1\dots j_\ell}(g, e, D),\\
&E_{;j_1\dots j_\ell}(g_c, e_c, D_c) = c^{2+\ell} E_{;j_1\dots j_\ell}(g, e, D)\,.
\end{aligned}$$

With this rescaling behavior in mind, define

$$\begin{aligned}
&\text{weight}\,(\phi_{;j_1\dots j_\ell}) = \ell, &&\text{weight}\,(\rho_{;j_1\dots j_\ell}) = \ell,\\
&\text{weight}\,(R_{i_1 i_2 i_3 i_4;j_1\dots j_\ell}) = 2+\ell, &&\text{weight}\,(L_{a_1 a_2:b_1\dots b_\ell}) = 1+\ell,\\
&\text{weight}\,(\Omega_{i_1 i_2;j_1\dots j_\ell}) = 2+\ell, &&\text{weight}\,(E_{;j_1\dots j_\ell}) = 2+\ell\,.
\end{aligned}$$

We may then use Theorem 2.1.12 (2) to see that $\beta_n^{\partial M}(\phi, \rho, D, \mathcal{B})$ is homogeneous of total weight $n-1$ in the variables listed in Equation (2.2.d). In other words, if A is a monomial term of $\beta_n^{\partial M}$ of degree $(k_R, k_L, k_\Omega, k_E)$ and if k_∇ explicit covariant derivatives appear, then

$$n - 1 = 2k_R + k_L + 2k_\Omega + 2k_E + k_\nabla \,.$$

We can now apply the first main theorem of invariance theory as discussed in Section 1.7.1 to see that $\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})$ can be expressed in terms of contractions of tangential indices ranging from 1 to $m-1$; the normal variable e_m is "free" since the structure group is $O(m-1)$. After taking into consideration the fact that β_n is bilinear in ϕ and ρ, the following Lemma is now immediate:

Lemma 2.2.12 *Let $\mathcal{B}$ define Dirichlet boundary conditions for an operator D of Laplace type.*

1. $\beta_0^{\partial M}(\phi,\rho,D,\mathcal{B}) = 0$.
2. $\beta_1^{\partial M}(\phi,\rho,D,\mathcal{B}) \in \operatorname{Span}\{\langle\phi,\rho\rangle\}$.
3. $\beta_2^{\partial M}(\phi,\rho,D,\mathcal{B}) \in \operatorname{Span}\{\langle\phi_{;m},\rho\rangle, \langle\phi,\rho_{;m}\rangle, L_{aa}\langle\phi,\rho\rangle\}$.
4. $\beta_3^{\partial M}(\phi,\rho,D,\mathcal{B}) \in \operatorname{Span}\{\langle\phi_{;mm},\rho\rangle, \langle\phi,\rho_{;mm}\rangle, \langle\phi_{;m},\rho_{;m}\rangle, \langle\phi_{:a},\rho_{:a}\rangle,$ $\langle E\phi,\rho\rangle, L_{aa}\langle\phi_{;m},\rho\rangle, L_{aa}\langle\phi,\rho_{;m}\rangle, \tau\langle\phi,\rho\rangle, R_{amma}\langle\phi,\rho\rangle,$ $L_{aa}L_{bb}\langle\phi,\rho\rangle, L_{ab}L_{ab}\langle\phi,\rho\rangle\}$.

The invariants $\langle\phi_{:aa},\rho\rangle$ and $\langle\phi,\rho_{:aa}\rangle$ have been omitted since

$$\int_{\partial M}\langle\phi_{:aa},\rho\rangle dy = \int_{\partial M}\langle\phi,\rho_{:aa}\rangle dy = -\int_{\partial M}\langle\phi_{:a},\rho_{:a}\rangle dy\,.$$

For the remaining boundary conditions, there are additional structures present. With Robin boundary conditions, $\mathcal{B}\phi = (\nabla_{e_m} + S)\phi$. We set $S_c := cS$ to express

$$\mathcal{B}_c\phi := c(\nabla_{e_m} + S)\phi = (\nabla_{ce_m} + S_c)\phi\,.$$

Clearly the different operators $\mathcal{B}$ and $\mathcal{B}_c$ define the same boundary condition. Since $S_c = cS$, we set

$$\text{weight}\,(S) = 1\,.$$

Taking into account this additional invariant, we have:

Lemma 2.2.13 *Let $\mathcal{B}$ define Robin boundary conditions for an operator D of Laplace type.*

1. $\beta_0^{\partial M}(\phi,\rho,D,\mathcal{B}) = 0$.
2. $\beta_1^{\partial M}(\phi,\rho,D,\mathcal{B}) \in \operatorname{Span}\{\langle\phi,\rho\rangle\}$.
3. $\beta_2^{\partial M}(\phi,\rho,D,\mathcal{B}) \in \operatorname{Span}\{\langle\phi_{;m},\rho\rangle, \langle\phi,\rho_{;m}\rangle, L_{aa}\langle\phi,\rho\rangle, \langle S\phi,\rho\rangle\}$.

With mixed boundary conditions, the auxiliary endomorphism χ and its covariant derivatives enter. Since χ has weight 0, a bit of care must be taken in writing down the relevant invariance theory and we postpone the discussion until the appropriate moment in Section 2.5.

With transmission boundary conditions we set weight$\,(U) = 1$, while with transfer boundary conditions, we set weight$\,(S_{\pm,\pm}) = 1$. Again, the weight is increased by 1 for every explicit covariant derivative that is present. The situation there is a bit more complicated since we have $\phi_\pm$, $\rho_\pm$, and so forth, so we shall again postpone the discussion of this case.

In studying β_4^M for any boundary condition, we will replace the interior integrand $\frac{1}{2}\langle D^2\phi,\rho\rangle$ by the interior integrand $\frac{1}{2}\langle D\phi,\tilde{D}\rho\rangle$ to maintain the symmetry of Lemma 2.1.3. This can be done, of course, at the cost of changing $\beta_4^{\partial M}$ to reflect the additional compensating boundary correction terms.

2.2.5 *Dimension shifting*

It is a very general principle that when the heat content asymptotics are expressed relative to a Weyl basis that the coefficients are universal expressions which are independent both of the dimension of the underlying manifold and also of the dimension of the vector bundle V. We illustrate this principle for Dirichlet boundary conditions.

Lemma 2.2.14 *There exist universal constants c_i which are independent of the dimension of the underlying manifold M and of the vector bundle V so that if $\mathcal{B}$ defines Dirichlet boundary conditions for an operator D of Laplace type, then:*

1. $\beta_1^{\partial M}(\phi,\rho,D,\mathcal{B}) = \int_{\partial M} c_1\langle\phi,\rho\rangle dy$.

2. $\beta_2^{\partial M}(\phi,\rho,D,\mathcal{B}) = \int_{\partial M}\{c_2\langle\phi_{;m},\rho\rangle + c_3\langle\phi,\rho_{;m}\rangle + c_3 L_{aa}\langle\phi,\rho\rangle\}dy$.

Proof: Let $m := \dim(M)$ and $r := \operatorname{Rank}(V)$. We use Lemma 2.2.12 to express β_1, β_2, and β_3 in the form given in the Lemma where a-priori the constants $c_i = c_i(m,r)$ depend upon the auxiliary parameters m and r. By Lemma 2.1.7,

$$\begin{aligned}
&\beta_n(\phi_1\oplus\phi_2,\rho_1\oplus\rho_2,D_1\oplus D_2,\mathcal{B}_1\oplus\mathcal{B}_2)\\
=\;&\beta_n(\phi_1,\rho_1,D_1,\mathcal{B}_1)+\beta_n(\phi_2,\rho_2,D_2,\mathcal{B}_2)\,.
\end{aligned}$$

It now follows that $c_i(m,r_1+r_2)=c_i(m,r_1)=c_i(m,r_2)$ and consequently

$$c_i(m,r)=c_i(m)\,.$$

Given the structure $(M_2,g_2,D_2,\mathcal{B}_2,\rho_2,\phi_2)$, we set $M := S^1\times M_2$ where S^1 is the unit circle. Give M the product metric

$$ds_M^2 := d\theta^2 + ds_{M_2}^2\,.$$

Let $D_1 = -\partial_\theta^2$ on the trivial line bundle. Set $\phi_1=\rho_1=1$. We adopt the notation of Lemma 2.1.8 setting

$$\begin{aligned}
&D := D_1\otimes\operatorname{Id}_2+\operatorname{Id}_1\otimes D_2 = -\partial_\theta^2 + D_2,\\
&\mathcal{B} := \mathcal{B}_2\,.
\end{aligned}$$

By Lemma 2.1.1,

$$\beta_q(\phi_1,\rho_1,D_1)(\theta)=\begin{cases}1 \text{ if } q=0,\\ 0 \text{ if } q>0.\end{cases}$$

We may therefore apply Lemma 2.1.8 to see

$$\beta_n(\phi,\rho,D,\mathcal{B})(\theta,x_2)=\beta_n(\phi_2,\rho_2,D_2,\mathcal{B}_2)(x_2)\,. \tag{2.2.e}$$

Let $\mathcal{R}_{n,m}$ be the set of all local formulae which are homogeneous of weight n in the variables defined in Equation (2.2.d); by Theorem 2.1.12, we can regard the heat content asymptotics as defining elements $\beta_{n,m}^{\partial M}\in\mathcal{R}_{n,m}$. We follow the discussion of Section 1.7.4 to define the restriction map

$$r:\mathcal{R}_{n,m}\to\mathcal{R}_{n,m-1}$$

and to use an appropriate generalization of Theorem 1.7.3 to see that this is surjective. Equation (2.2.e) then shows $r(\beta_{n,m}^{\partial M}) = \beta_{n,m-1}^{\partial M}$ and thus it is not in fact necessary to introduce the additional subscript m. The assertion that the constants c_i can be chosen independent of the dimension m now follows from this discussion. $\square$

The situation is the same for pure Neumann boundary conditions. For Robin, mixed, transmission, and transfer boundary conditions, additional invariants must be introduced which reflect the auxiliary structures which are involved. With oblique boundary conditions, the leading symbol of the tangential operator has weight 0 and more care needs to be exercised.

2.2.6 Spectral boundary conditions

Let P be an operator of Dirac type on a vector bundle V over a compact Riemannian manifold with boundary. Let A be an operator of Dirac type on $V|_{\partial M}$ which is admissible with respect to P; we refer to the discussion in Section 1.6.6 for details. Let $D = P^2$ be the associated operator of Laplace type and let $\mathcal{B}_A$ be the associated spectral boundary conditions for D. Let $\beta(\phi, \rho, D, \mathcal{B}_A)(t)$ be the associated heat content asymptotics. There is a $\mathbb{Z}_2$ symmetry involved using the involution $P \to -P$. The following observation is immediate:

Lemma 2.2.15 *Let A be admissible with respect to P. Then A is admissible with respect to $-P$ and $\beta(\phi, \rho, P^2, \mathcal{B}_A)(t) = \beta(\phi, \rho, (-P)^2, \mathcal{B}_A)(t)$.*

The next functorial property relates the heat content asymptotics for spectral boundary conditions to the corresponding heat content asymptotics for mixed boundary conditions in certain settings. We recall the notation of Section 1.6.6. Let P be an operator of Dirac type on a vector bundle V and let A be an admissible operator of Dirac type on $V|_{\partial M}$ which defines the boundary conditions Π_A^+ for P. Let $x = (y_1, ..., y_{m-1}, x_m)$ be normalized coordinates near the boundary. We may expand P and A locally in the form

$$P = \gamma_i \nabla_{e_i} + \psi_P \qquad \text{and} \qquad A = -\gamma_m \gamma_a \nabla_{e_a} + \psi_A$$

where ψ_P and ψ_A are 0^{th} order terms and where ∇ is a compatible connection. We use Lemma 1.6.7 to see that (P, Π_A^+) satisfies a suitable ellipticity condition and thus the heat content asymptotics are well defined for the associated second order operator and boundary condition

$$D := P^2 \qquad \text{and} \qquad \mathcal{B}_A := \Pi_A^+ \oplus \Pi_A^+ P\,.$$

In the next result, we will relate spectral and mixed boundary conditions. Before giving the statement, it is necessary to introduce some notation. Let $(\theta_1, ..., \theta_{m-1})$ be the usual periodic parameters on the torus $\mathbb{T}^{m-1}$ and let r be the radial parameter on the interval $[0,1]$. Give $M := \mathbb{T}^{m-1} \times [0,1]$ a warped product of the form

$$ds_M^2 := g_{ab}(r) d\theta^a \circ d\theta^b + dr^2$$

and set $g(r) := \sqrt{\det g_{ab}}$. Let $\varepsilon(0) := +1$ and $\varepsilon(1) := -1$ so that the inward unit normal is εdr on

$$\partial M = \mathbb{T}^{m-1} \times \{\{0\} \cup \{1\}\}.$$

Let $\{\gamma_1(r), ..., \gamma_m(r)\}$ give $V := M \times \mathbb{C}^\ell$ a Clifford module structure; we suppose

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab}\mathrm{Id}, \quad \gamma_a\gamma_m + \gamma_m\gamma_a = 0, \quad \text{and} \quad \gamma_m^2 = -\mathrm{Id}.$$

We set $\gamma^a := g^{ab}(r)\gamma_a$. Let $p_0 = p_0(r)$ and $a_0 = a_0(r)$ be $\ell \times \ell$ matrices; we suppose a_0 is self-adjoint and invertible. Define operators of Dirac type on the trivial bundle of dimension ℓ by setting

$$\begin{aligned}
&P_0 := \gamma_m\partial_r + p_0 \quad \text{on} \quad C^\infty([0,1] \times \mathbb{C}^\ell),\\
&P := P_0 + \gamma^a\partial_a^\theta \quad \text{on} \quad C^\infty(M \times \mathbb{C}^\ell),\\
&A := \varepsilon(-\gamma_m\gamma^a\partial_a^\theta + a_0) \quad \text{on} \quad \partial M.
\end{aligned}$$

We assume $\ker(A) = \{0\}$. Then A is admissible for P and we let Π_A^+ and $\mathcal{B}_A$ be the associated boundary conditions for P and for $D := P^2$, respectively.

Let Π_0 be orthogonal projection on the positive eigenspaces of a_0 and let $\Pi_1 = \mathrm{Id} - \Pi_0$ be the complementary projection. We assume a_0 anti-commutes with γ_m. We then have $\gamma_m\Pi_1 = \Pi_0\gamma_m$. Let

$$\mathcal{B}_0\phi := \{\Pi_0\phi \oplus \Pi_0 P_0\phi\}|_{\partial[0,1]}.$$

We show that $\mathcal{B}_0$ defines mixed boundary conditions for $D_0 := P_0^2$ by considering the following equivalent formulation. We have

$$\begin{aligned}
&\Pi_0(\phi|_{\partial M}) = 0 \quad \text{and} \quad \Pi_0(\gamma_m\partial_r\phi + p_0)|_{\partial M} = 0,\\
\Leftrightarrow\ &\Pi_0(\phi|_{\partial M}) = 0 \quad \text{and} \quad \Pi_0\gamma_m(\partial_r\phi - \gamma_m p_0)|_{\partial M} = 0,\\
\Leftrightarrow\ &\Pi_0(\phi|_{\partial M}) = 0 \quad \text{and} \quad \gamma_m\Pi_1(\partial_r\phi - \gamma_m p_0)|_{\partial M} = 0,\\
\Leftrightarrow\ &\Pi_0(\phi|_{\partial M}) = 0 \quad \text{and} \quad \Pi_1(\partial_r\phi - \gamma_m p_0)|_{\partial M} = 0.
\end{aligned}$$

Lemma 2.2.16 *Adopt the notational conventions and assumptions described above. Let $\phi = \phi(r)$ and $\rho = \rho(r)$ be smooth $\mathbb{C}^\ell$ and $(\mathbb{C}^\ell)^*$ valued functions, respectively. Note that $\Pi_a^+\phi = \Pi_0\phi$.*

1. $\beta(\phi, g^{-1}\rho, D, \mathcal{B}_A)(t) = (2\pi)^{m-1}\beta(\phi, \rho, D_0, \mathcal{B}_0)(t)$.

2. $\beta_n(\phi, g^{-1}\rho, D, \mathcal{B}_A) = (2\pi)^{m-1}\beta_n(\phi, \rho, D_0, \mathcal{B}_0)$.

Proof: Let $u_0 := e^{-tD_{0,\mathcal{B}_0}}\phi$. Set $u(r,\theta;t) := u_0(r;t)$. We show $u = e^{-tD_{\mathcal{B}}}\phi$ by checking that the defining relations are satisfied

$$\begin{aligned}
&(\partial_t + D)u = (\partial_t + D_0)u = 0,\\
&\mathcal{B}u = \Pi_A^+ u \oplus \Pi_A^+ Pu = \Pi_0 u \oplus \Pi_0 P_0 u = 0,\\
&u|_{t=0} = u_0|_{t=0} = \phi.
\end{aligned}$$

Assertion (1) now follows as ρ has been adjusted to take into account the change in the volume element. We equate terms in the asymptotic expansions to derive Assertion (2) from Assertion (1). $\square$

The analogue of Lemma 2.2.14 requires a different proof since we do not have product formulae available. It is somewhat surprising that the coefficients with respect to a suitable Weyl basis are dimension free. The corresponding coefficients for the heat trace asymptotics exhibit a very complicated dependence on the dimension as we shall see in Section 3.14. We refer to the subsequent discussion in Ansatz 2.12.3 for a more detailed discussion of the sorts of expressions that occur and content ourselves here with a quite general statement.

Lemma 2.2.17 *Let P be an operator of Dirac type on a bundle V over a compact manifold M with smooth boundary ∂M. Let A be admissible with respect to P. Express β_n with respect to a suitable Weyl basis. Then the universal coefficients that appear are independent of the rank of the bundle and the dimension of the manifold.*

Proof: The proof of Lemma 2.1.7 shows the coefficients are independent of the rank of the bundle. Let P_N be an operator of Dirac type on an $m-1$ dimensional manifold N. Give

$$M := S^1 \times N$$

the product Riemannian metric. We introduce coordinates (θ, x) on M. By doubling the rank of the vector bundle and by replacing P_0 by $P_N \oplus -P_N$ if necessary, we may suppose there exists γ_0 so that

$$\gamma_0 P_N + P_N \gamma_0 = 0 \quad \text{and} \quad \gamma_0^2 = -\mathrm{Id}\ .$$

Define analogous structures on M by setting

$$P := \gamma_0 \partial_\theta + P_N \qquad \text{and} \qquad A := -\gamma_m \gamma_0 \partial_\theta + A_N\ .$$

Let $\phi(x, \theta) = \phi_N(x)$. Then

$$\begin{aligned} &P\phi = P_N \phi_N, && A\phi = A_N \phi_N, \\ &D\phi = D_N \phi_N, && (\Pi_A^+ \phi)(x,\theta) = (\Pi_{A_N}^+ \phi_N)(x), \\ &(\mathcal{B}_A \phi)(x,\theta) = (\mathcal{B}_{A_N} \phi_N)(x)\ . \end{aligned}$$

Let $u_N := e^{-tD_{N,\mathcal{B}_N}} \phi_N$ and let $u(x,\theta;t) := u_N(x;t)$. We show $u = e^{-tD_\mathcal{B}} \phi$ by verifying the defining relations are satisfied

$$\begin{aligned} &(\partial_t + D)u = (\partial_t + D_N)u_N = 0, \\ &\mathcal{B}u = \mathcal{B}_N u_N = 0, \\ &u|_{t=0} = u_N|_{t=0} = \phi\ . \end{aligned}$$

It now follows that

$$\begin{aligned} &\beta(\phi, \rho, D, \mathcal{B}_A)(t) = 2\pi \beta(\phi_N, \rho_N, D_N, \mathcal{B}_N)(t) \quad \text{so} \\ &\beta_n(\phi, \rho, D, \mathcal{B}_A) = 2\pi \beta_n(\phi_N, \rho_N, D_N, \mathcal{B}_N)\ . \end{aligned}$$

The desired independence of the universal constants now follows. $\square$

2.3 Heat content asymptotics for Dirichlet boundary conditions

Throughout this section, let $\mathcal{B}\phi := \phi|_{\partial M}$ define *Dirichlet boundary conditions.* The case $\phi = \rho = 1$ of constant initial temperature and specific heat and an evolution equation determined by the scalar Laplacian $D = \Delta^0$ are of particular interest and we shall set

$$\beta_n(M) := \beta_n(1, 1, \Delta^0, \mathcal{B}).$$

We begin with two special case computations which are of interest in their own right and which deal with this situation. Recall the definition of the *Gamma function*

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt.$$

We first discuss the disk in flat space and give results of [46]:

Theorem 2.3.1 *Let D^m be the unit disk in $\mathbb{R}^m$. Then:*

1. $\beta_0(D^m) = \frac{\pi^{m/2}}{\Gamma(\frac{(2+m)}{2})}$.
2. $\beta_1(D^m) = -4\frac{\pi^{(m-1)/2}}{\Gamma(\frac{m}{2})}$.
3. $\beta_2(D^m) = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2})}(m-1)$.
4. $\beta_3(D^m) = -\frac{\pi^{(m-1)/2}}{3\Gamma(\frac{m}{2})}(m-1)(m-3)$.
5. $\beta_4(D^m) = -\frac{\pi^{m/2}}{8\Gamma(\frac{m}{2})}(m-1)(m-3)$.
6. $\beta_5(D^m) = \frac{\pi^{(m-1)/2}}{120\Gamma(\frac{m}{2})}(m-1)(m-3)(m+3)(m-7)$.
7. $\beta_6(D^m) = \frac{\pi^{m/2}}{96\Gamma(\frac{m}{2})}(m-1)(m-3)(m^2-4m-9)$.
8. $\beta_7(D^m) = -\frac{\pi^{(m-1)/2}}{3360\Gamma(\frac{m}{2})}(m-1)(m-3)(m^4-8m^3-90m^2+424m+633)$.

Next, we present some results for the hemisphere from [47]:

Theorem 2.3.2 *Let H^m be the upper hemisphere of the unit sphere S^m. Then*

1. $\beta_{2k}(H^m) = 0$ *for any m if $k > 0$.*
2. $\beta_{2k+1}(H^3) = \frac{8\pi^{1/2}}{k!(2k-1)(2k+1)}$.
3. $\beta_{2k+1}(H^5) = \frac{\pi^{3/2}2^{2k+3}(2-k)}{3k!(2k-1)(2k+1)}$.
4. $\beta_{2k+1}(H^7) = \frac{\pi^{5/2}}{30}\left\{\frac{(67-54k)9^k}{k!(2k-1)(2k+1)} + \sum_{\ell=0}^k \frac{3\cdot 2^{3\ell}}{\ell!(k-\ell)!(2k-2\ell+1)}\right\}$.

The following general result [45, 46] is the main result of this section:

Theorem 2.3.3 *Let D be an operator of Laplace type on a compact Riemannian manifold with smooth boundary ∂M. Let $\mathcal{B}\phi = \phi|_{\partial M}$ define Dirichlet boundary conditions. Then*

1. $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle dx$.

2. $\beta_1(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\phi,\rho\rangle dy$.

3. $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\{\langle\frac{1}{2}L_{aa}\phi,\rho\rangle - \langle\phi,\rho_{;m}\rangle\}dy$.

4. $\beta_3(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\{\frac{2}{3}\langle\phi_{;mm},\rho\rangle + \frac{2}{3}\langle\phi,\rho_{;mm}\rangle - \langle\phi_{:a},\rho_{:a}\rangle$
$+\langle E\phi,\rho\rangle - \frac{2}{3}L_{aa}\langle\phi,\rho\rangle_{;m} + \langle(\frac{1}{12}L_{aa}L_{bb} - \frac{1}{6}L_{ab}L_{ab} - \frac{1}{6}R_{amma})\phi,\rho\rangle\}dy$.

5. $\beta_4(\phi,\rho,D,\mathcal{B}) = \frac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle dx + \int_{\partial M}\{\frac{1}{2}\langle(D\phi)_{;m},\rho\rangle + \frac{1}{2}\langle\phi,(\tilde{D}\rho)_{;m}\rangle$
$-\frac{1}{4}\langle L_{aa}D\phi,\rho\rangle - \frac{1}{4}\langle L_{aa}\phi,\tilde{D}\rho\rangle + \langle(\frac{1}{8}E_{;m} - \frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac{1}{8}L_{ab}L_{ac}L_{bc}$
$-\frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac} + \frac{1}{32}\tau_{;m} + \frac{1}{16}L_{ab:ab})\phi,\rho\rangle$
$-\frac{1}{4}L_{ab}\langle\phi_{:a},\rho_{:b}\rangle - \frac{1}{8}\langle\Omega_{am}\phi_{:a},\rho\rangle + \frac{1}{8}\langle\Omega_{am}\phi,\rho_{:a}\rangle\}dy$.

We set $\phi=\rho=1$ and $E=\Omega=0$ in Theorem 2.3.3 to compute $\beta_n(M)$ for $n\le 4$. The invariant $\beta_5(M)$ is known [49], although $\beta_5(\phi,\rho,D,\mathcal{B})$ is not known in full generality:

Theorem 2.3.4 *Let $\phi=\rho=1$ and let $D=\Delta^0$ be the scalar Laplacian on a compact manifold M with smooth boundary ∂M. Let $\mathcal{B}$ denote Dirichlet boundary conditions. Then*

$$\begin{aligned}\beta_5(1,1,\Delta^0,\mathcal{B}) = -\tfrac{1}{240\sqrt{\pi}}\int_{\partial M}\{&8\rho_{mm;mm} - 8L_{aa}\rho_{mm;m} + 16L_{ab}R_{ammb;m}\\ &-4\rho_{mm}^2 + 16R_{ammb}R_{ammb} - 4L_{aa}L_{bb}\rho_{mm} - 8L_{ab}L_{ab}\rho_{mm}\\ &+64L_{ab}L_{ac}R_{mbcm} - 16L_{aa}L_{bc}R_{mbcm} - 8L_{ab}L_{ac}R_{bddc}\\ &-8L_{ab}L_{cd}R_{acbd} + 4R_{abcm}R_{abcm} + 8R_{abbm}R_{accm} - 16L_{aa:b}R_{bccm}\\ &-8L_{ab:c}L_{ab:c} + L_{aa}L_{bb}L_{cc}L_{dd} - 4L_{aa}L_{bb}L_{cd}L_{cd} + 4L_{ab}L_{ab}L_{cd}L_{cd}\\ &-24L_{aa}L_{bc}L_{cd}L_{db} + 48L_{ab}L_{bc}L_{cd}L_{da}\}dy\,.\end{aligned}$$

If M is a domain in $\mathbb{R}^m$, then $\beta_0(M)$ and $\beta_1(M)$ were computed by van den Berg and Davies [44]; $\beta_2(M)$ was subsequently computed by van den Berg and Le Gall [53]. If M is the upper hemisphere of the unit sphere in flat space, then $\beta_0(M)$, $\beta_1(M)$, and $\beta_2(M)$ were computed by van den Berg [37]; we refer to [47] for later work which establishes Theorem 2.3.2. There are related results due to Phillips and Janson [307]. Subsequently, Savo [331, 333, 334, 335] gave a closed formula for all the heat content asymptotics $\beta_k(M)$ in this setting. We also refer to related work by McDonald and Meyers [276].

The general case where D is an arbitrary operator of Laplace type and where ϕ and ρ are quite arbitrary was studied in [45, 46] and we shall follow the treatment there in discussing Theorem 2.3.3. We also refer to McAvity [272] for a slightly different approach using the DeWitt ansatz.

The remainder of this section is devoted to the proof of Assertions (1), (2), (3), and (4) of Theorem 2.3.3. Since Theorem 2.3.3 (5) follows by similar techniques, we shall simply refer to [45] in the interest of brevity.

Assertion (1) of Theorem 2.3.3 follows from Lemma 2.1.1. We shall use the arguments used to established Lemma 2.2.14 to see that we may express

β_n as the integral of local invariants with certain undetermined universal coefficients c_i. We shall complete the proof of Theorem 2.3.3 by determining these universal coefficients.

We begin our investigation by determining β_1 in this setting:

Lemma 2.3.5 *Let $\mathcal{B}$ be the Dirichlet boundary operator. Then*

$$\beta_1(\phi,\rho,D,\mathcal{B}) = -\tfrac{2}{\sqrt{\pi}}\textstyle\int_{\partial M}\langle\phi,\rho\rangle dy\,.$$

Proof: By Lemma 2.2.14, there exists a universal constant so

$$\beta_1(\phi,\rho,D,\mathcal{B}) = c_1\int_{\partial M}\langle\phi,\rho\rangle dy\,.$$

We apply the *method of universal examples* to determine c_1. We use Example 1.4.4. Let $M=[0,\pi]$ be the interval, let $D=-\partial_x^2$, and let $\rho=\phi=1$. Since the structures are flat on $[0,\pi]$,

$$\beta(t)\sim\pi+2c_1\sqrt{t}+O(t)\,. \tag{2.3.a}$$

The spectral resolution of the Laplacian with Dirichlet boundary conditions on the interval is given by

$$\left\{\frac{\sqrt{2}}{\sqrt{\pi}}\sin nx, n^2\right\}_{n=1}^{\infty}\,.$$

Consequently, the associated Fourier coefficients are

$$\sigma_n(\phi)=\sigma_n(\rho)=\tfrac{\sqrt{2}}{\sqrt{\pi}}\int_0^\pi\sin(nx)dx=\tfrac{\sqrt{2}}{\sqrt{\pi}}\begin{cases}\frac{2}{n} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Therefore, we may use Lemma 2.1.6 to see that

$$\beta(\phi,\rho,D,\mathcal{B})(t)=\sum_{n=1}^{\infty}e^{-tn^2}\sigma_n(\phi)\sigma_n(\rho)=\tfrac{8}{\pi}\sum_{n-\mathrm{odd}}\tfrac{1}{n^2}e^{-tn^2}\,. \tag{2.3.b}$$

Differentiating Equations (2.3.a) and (2.3.b) then yields

$$\partial_t\beta(\phi,\rho,D,\mathcal{B})=-\tfrac{8}{\pi}\sum_{n-\mathrm{odd}}e^{-tn^2}\sim c_1t^{-1/2}+O(1)\,.$$

We rewrite this identity slightly to see

$$\lim_{t\downarrow 0}\;2\sqrt{t}\cdot\sum_{n-\mathrm{odd}}e^{-\{\sqrt{t}\cdot n\}^2}\sim-\tfrac{\pi}{4}c_1\,. \tag{2.3.c}$$

We consider the function $f(x):=e^{-x^2}$ and let $\delta:=\sqrt{t}$. We use Riemann sums with a width of 2δ and center points $(2k+1)\delta$ for $k=0,1,...$ to compute the area under the curve

$$\begin{aligned}\tfrac{\sqrt{\pi}}{2} &= \int_0^\infty e^{-x^2}dx \qquad (2.3.d)\\ &= \lim_{\delta\downarrow 0}\;2\delta\Big\{f(\delta)+f(3\delta)+f(5\delta)+...\Big\}\end{aligned}$$

$$= \lim_{t\downarrow 0} 2\sqrt{t}\cdot \sum_{n-\text{odd}} e^{-\{\sqrt{t}\cdot n\}^2}.$$

We use Equations (2.3.c) and (2.3.d) to complete the proof by checking

$$c_1 = -\tfrac{2}{\sqrt{\pi}}. \qquad \square$$

Next we study β_2. By Lemma 2.2.14, there exist universal constants so that

$$\begin{aligned}\beta_2(\phi,\rho,D,\mathcal{B}) &= -\int_M \langle D\phi,\rho\rangle dx \\ &+ \int_{\partial M}\Big\{c_2\langle\phi_{;m},\rho\rangle + c_3\langle\phi,\rho_{;m}\rangle + c_4 L_{aa}\langle\phi,\rho\rangle\Big\}dy\,.\end{aligned}$$

Assertion (3) of Theorem 2.3.3 will follow from the following result:

Lemma 2.3.6 $c_2 = 0$, $c_3 = -1$, $c_4 = \frac{1}{2}$.

Proof: Suppose that $\phi|_{\partial M} = 0$. We may then apply Lemma 2.1.4 to equate

$$\begin{aligned}&\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M \langle D\phi,\rho\rangle dx + c_2\int_{\partial M}\langle\phi_{;m},\rho\rangle dy \\ =\; & -\beta_0(D\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle dx\,.\end{aligned}$$

By Lemma 1.4.1, we can choose ϕ so that $\phi|_{\partial M} = 0$ and so that $\phi_{;m}|_{\partial M}$ is arbitrary. Therefore

$$c_2 = 0\,.$$

By Lemma 1.5.1, the Dirichlet boundary operator is the adjoint boundary operator. We use Lemma 2.1.3 to equate

$$\begin{aligned}&\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\Big\{c_3\langle\phi,\rho_{;m}\rangle + c_4L_{aa}\langle\phi,\rho\rangle\Big\}dy \\ =\; & \beta_2(\rho,\phi,\tilde{D},\tilde{\mathcal{B}}) = -\int_M\langle\tilde{D}\rho,\phi\rangle dx + \int_{\partial M}\Big\{c_3\langle\rho,\phi_{;m}\rangle + c_4L_{aa}\langle\rho,\phi\rangle\Big\}dy\,.\end{aligned}$$

This identity implies that

$$\int_M\{\langle D\phi,\rho\rangle - \langle\phi,\tilde{D}\rho\rangle\}dx = c_3\int_{\partial M}\Big\{\langle\phi,\rho_{;m}\rangle - \langle\phi_{;m},\rho\rangle\Big\}dy\,.$$

The Green's formula given in Lemma 1.4.17 then shows that

$$c_3 = -1\,.$$

Let $ds^2 := dr^2 + e^{2f(r)}d\theta^2$ be a warped product metric on $M := S^1\times[0,1]$. Relative to the coordinate frame $\{\partial_\theta,\partial_r\}$, the non-zero Christoffel symbols are

$$\Gamma_{\theta r\theta} = \Gamma_{r\theta\theta} = -\Gamma_{\theta\theta r} = \partial_r f e^{2f}\,.$$

We shall suppose that $f(0) = 0$ and that f vanishes identically near $r = 1$.

Thus only the boundary component $r = 0$ will be relevant; on this boundary component, ∂_r will be the inward unit normal. Let

$$D_1 := -\partial_r^2, \quad D_2 := -e^{-2f(r)}\partial_\theta^2, \quad \text{and} \quad D := -\partial_r^2 - e^{-2f(r)}\partial_\theta^2 .$$

Since the structures on the interval are flat, $\beta_2(1, 1, D_1, \mathcal{B}) = 0$. We may therefore use Lemma 2.1.9 with $\phi = 1$ and $\rho = e^{-f}$ to see that

$$\beta_2(1, e^{-f}, D, \mathcal{B}) = 2\pi\beta_2(1, 1, D_1, \mathcal{B}) = 0 . \tag{2.3.e}$$

By Lemma 1.2.1, $\omega_\mu = \frac{1}{2}(g_{\mu\nu}a^\mu + g^{\nu\sigma}\Gamma_{\nu\sigma\mu})$. Thus we have that

$$\omega_\theta = 0 \quad \text{and} \quad \omega_r = -\tfrac{1}{2}\partial_r f .$$

We use the fact that $\tilde{\omega}_\mu = -\omega_\mu$ to compute

$$\begin{aligned}
&\rho_{;m}|_{r=0} = \{(\partial_r + \tfrac{1}{2}\partial_r f)e^{-f}\}|_{r=0} = -\tfrac{1}{2}\partial_r f(0), \\
&L_{aa}|_{r=0} = \Gamma_{\theta\theta r} = -\partial_r f(0), \\
&\beta_2(1, e^{-f}, D, \mathcal{B}) = 2\pi(-\tfrac{1}{2}c_3 - c_4)\partial_r f(0) .
\end{aligned}$$

Thus by Equation (2.3.e), we have

$$-\tfrac{1}{2}c_3 - c_4 = 0 \quad \text{so} \quad c_4 = -\tfrac{1}{2}c_3 = \tfrac{1}{2}. \qquad \square$$

The proof of Theorem 2.3.3 (4) is similar. There is no interior integrand so by Lemma 2.1.3, the boundary integrands are symmetric in ϕ and ρ. We include the normalizing constant $-\frac{2}{\sqrt{\pi}}$ involved in β_1 to simplify subsequent computation and use the results of Section 2.2.4 to express β_3 in terms of a Weyl basis. There are universal constants c_i so that

$$\begin{aligned}
\beta_3(\phi, \rho, D, \mathcal{B}) = -\tfrac{2}{\sqrt{\pi}} \int_{\partial M} \Big\{ & c_5\langle E\phi, \rho\rangle + c_6(\langle \phi_{;mm}, \rho\rangle + \langle \phi, \rho_{;mm}\rangle) \\
& + c_7 L_{aa}\langle \phi, \rho\rangle_{;m} + c_8 R_{abba}\langle \phi, \rho\rangle + c_9\langle \phi_{:a}, \rho_{:a}\rangle \\
& + (c_{10}R_{amma} + c_{11}L_{aa}L_{bb} + c_{12}L_{ab}L_{ab})\langle \phi, \rho\rangle \\
& + c_{13}\langle \phi_{;m}, \rho_{;m}\rangle \Big\} dy .
\end{aligned} \tag{2.3.f}$$

Theorem 2.3.3 (4) will follow from the following result:

Lemma 2.3.7

1. $c_5 = 1$.

2. $c_6 = \frac{2}{3}$, $c_7 = -\frac{2}{3}$, *and* $c_{13} = 0$.

3. $c_8 = 0$.

4. $c_9 = -1$.

5. $c_{10} = -\frac{1}{6}$, $c_{11} = \frac{1}{12}$, *and* $c_{12} = -\frac{1}{6}$.

Proof: By Lemma 2.1.2 and Theorem 2.3.3 (2),

$$\partial_\varepsilon\beta_3(\phi, \rho, D - \varepsilon \cdot \mathrm{Id}, \mathcal{B}) = \beta_1(\phi, \rho, D, \mathcal{B}) = -\tfrac{2}{\sqrt{\pi}} \int_{\partial M} \langle \phi, \rho\rangle dy . \tag{2.3.g}$$

Since $E(D-\varepsilon\cdot\mathrm{Id}\,)=E(D)+\varepsilon\cdot\mathrm{Id}$, we have by Equation (2.3.f) that

$$\partial_\varepsilon\beta_3(\phi,\rho,D-\varepsilon\cdot\mathrm{Id}\,,\mathcal{B})=-\tfrac{2}{\sqrt{\pi}}\int_{\partial M}c_5\langle E\phi,\rho\rangle dy\,. \tag{2.3.h}$$

We use Equations (2.3.g) and (2.3.h) to derive Assertion (1) by checking that

$$c_5=1\,.$$

Suppose $\mathcal{B}\phi=0$. We use Lemma 1.1.4 to see

$$\phi_{;aa}=\phi_{:aa}-L_{aa}\phi_{;m}=-L_{aa}\phi_{;m}\quad\text{on}\quad\partial M\,.$$

Thus by Lemma 2.1.4 and Theorem 2.3.3 (2),

$$\begin{aligned}\beta_3(\phi,\rho,D,\mathcal{B})&=-\tfrac{2}{3}\beta_1(D\phi,\rho,D,\mathcal{B})\\&=(-\tfrac{2}{3})(-\tfrac{2}{\sqrt{\pi}})\int_{\partial M}\langle-\phi_{;mm}-\phi_{;aa},\rho\rangle dy\\&=-\tfrac{2}{\sqrt{\pi}}\int_{\partial M}\left\{\tfrac{2}{3}\langle\phi_{;mm},\rho\rangle-\tfrac{2}{3}L_{aa}\langle\phi_{;m},\rho\rangle\right\}dy\,.\end{aligned} \tag{2.3.i}$$

On the other hand, by Equation (2.3.f), one has that

$$\begin{aligned}\beta_3(\phi,\rho,D,\mathcal{B})\;&=\;-\tfrac{2}{\sqrt{\pi}}\int_{\partial M}\Big\{c_6\langle\phi_{;mm},\rho\rangle+c_7L_{aa}\langle\phi_{;m},\rho\rangle\\&+\;c_{13}\langle\phi_{;m},\rho_{;m}\rangle\Big\}dy\,.\end{aligned} \tag{2.3.j}$$

Assertion (2) now follows from Equations (2.3.i) and (2.3.j).

We use product formulae to establish the remaining assertions. To prove Assertion (3), let M_1 be an arbitrary closed Riemannian manifold. Give the manifold $M:=N\times[0,1]$ the product metric. Let D, D_1, and D_2 be the scalar Laplacians on M, M_1, and $[0,1]$, respectively. Let $\phi=\rho=1$. Since the structures on the interval are flat,

$$\beta_n(\phi,\rho,D_2,\mathcal{B}_2)=0\quad\text{for}\quad n>0\,.$$

As the boundary of M_1 is empty, Lemma 2.1.1 implies

$$\beta_n(\phi,\rho,D_2)=0\quad\text{for}\quad n>0\,.$$

Consequently, Lemma 2.1.8 shows that

$$\beta_3(\rho,\phi,D,\mathcal{B})=0\,.$$

Because $\beta_3(\rho,\phi,D,\mathcal{B})=-\frac{2}{\sqrt{\pi}}\int_{\partial M}c_8R_{abba}\langle\phi,\rho\rangle dy$, Assertion (3) follows as

$$c_8=0\,.$$

If (r,θ) are the usual parameters on $M:=S^1\times[0,1]$, let

$$D:=-\partial_r^2-\partial_\theta^2-2\varepsilon\partial_\theta,\quad\phi:=1,\quad\rho:=1\,.$$

Since the structures on the interval are flat, Lemma 2.1.8 implies that

$$\beta_3(\phi,\rho,D,\mathcal{B})=2\pi\beta_3(\phi,\rho,-\partial_r^2,\mathcal{B})=0\,. \tag{2.3.k}$$

For the structures defined by D, we use Lemma 1.2.1 to see that

$$\nabla_{\partial_r}\phi = 0, \quad \nabla_{\partial_r}\rho = 0, \quad \nabla_{\partial_\theta}\phi = \varepsilon\phi, \quad \nabla_{\partial_\theta}\rho = -\tilde{\varepsilon}\rho, \quad E = -\varepsilon^2 \,. \tag{2.3.l}$$

Therefore, Equation (2.3.f) implies that

$$\beta_3(\phi,\rho,D,\mathcal{B}) = \int_{\partial M}\left\{ -\varepsilon^2 c_5 - \varepsilon^2 c_9\right\} dy\,. \tag{2.3.m}$$

Assertion (4) follows from Equations (2.3.k) and (2.3.m) and Assertion (1) as

$$c_5 + c_9 = 0 \quad \text{so} \quad c_9 = -c_5 = -1\,.$$

To prove the final assertion, we consider the warped product metric

$$ds_M^2 := dr^2 + e^{2f_1(r)}d\theta_1^2 + e^{2f_2(r)}d\theta_2^2 \tag{2.3.n}$$

on the cylinder $M := \mathbb{T}^2 \times [0,1]$. As the inward unit normal is given by ∂_r when $r = 0$ and by $-\partial_r$ when $r = 1$, we shall assume that for $a = 1, 2$

$$\begin{aligned} &f_a(0) = 0, \quad \partial_r f_a(0) = \ \ \varepsilon_a, \quad \partial_r^2 f_a(0) = \varrho_a \\ &f_a(1) = 0, \quad \partial_r f_a(1) = -\varepsilon_a, \quad \partial_r^2 f_a(1) = \varrho_a\,. \end{aligned}$$

Relative to the coordinate frame $\{X_1 := \partial_{\theta_1}, X_2 := \partial_{\theta_2}, X_3 := \partial_r\}$,

$$\begin{aligned} &\Gamma_{311} = \Gamma_{131} = -\Gamma_{113} = \partial_r f_1 e^{2f_1}, \quad \text{and} \\ &\Gamma_{322} = \Gamma_{232} = -\Gamma_{223} = \partial_r f_2 e^{2f_2}\,; \end{aligned}$$

the remaining Christoffel symbols vanish. Since ∂_r is the inward unit normal when $r = 0$ and $-\partial_r$ is the inward unit normal when $r = 1$,

$$L_{ab} = -\varepsilon_a\delta_{ab}\,. \tag{2.3.o}$$

Raising indices then yields

$$\begin{aligned} &\Gamma_{11}{}^3 = -\partial_r f_1 e^{2f_1}, \quad \Gamma_{31}{}^1 = \partial_r f_1 \\ &\Gamma_{22}{}^3 = -\partial_r f_2 e^{2f_2}, \quad \Gamma_{32}{}^2 = \partial_r f_2\,. \end{aligned}$$

We apply Equation (1.1.a) to see

$$\begin{aligned} &R_{3113} = g_{33}(\partial_3\Gamma_{11}{}^3 - \Gamma_{11}{}^3\Gamma_{31}{}^1) = (-\partial_r^2 f_1 - 2\partial_r f_1\partial_r f_1 + \partial_r f_1\partial_r f_1)e^{2f_1}, \\ &R_{3223} = g_{33}(\partial_3\Gamma_{22}{}^3 - \Gamma_{22}{}^3\Gamma_{32}{}^2) = (-\partial_r^2 f_2 - 2\partial_r f_2\partial_r f_2 + \partial_r f_2\partial_r f_2)e^{2f_2}, \\ &R_{amma} = -\varrho_1 - \varrho_2 - \varepsilon_1^2 - \varepsilon_2^2 \quad \text{on} \quad \partial M\,. \end{aligned}$$

Let

$$D_1 := -\partial_r^2 \quad \text{and} \quad D := -\partial_r^2 - e^{-2f_1(r)}\partial_{\theta_1}^2 - e^{-2f_2(r)}\partial_{\theta_2}^2\,.$$

Let $\phi = 1$ and let $\rho := e^{-f_1(r)-f_2(r)}$. We apply Lemma 2.1.9 twice to see

$$\begin{aligned} \beta_3(\phi,\rho,D,\mathcal{B}) &= 2\pi\beta_3(\phi, e^{f_2}\rho, -\partial_r^2 - e^{-2f_2}\partial_{\theta_2}^2, \mathcal{B}), \\ &= (2\pi)^2\beta_3(\phi, e^{f_1+f_2}\rho, -\partial_r^2, \mathcal{B}), \\ &= (2\pi)^2\beta_3(1,1,-\partial_r^2,\mathcal{B}) = 0\,. \end{aligned}$$

We use Lemma 1.2.1 to compute

$$\begin{aligned}
&\omega_3 = \tfrac{1}{2}(g^{11}\Gamma_{113} + g^{22}\Gamma_{223}) = -\tfrac{1}{2}(\partial_r f_1 + \partial_r f_2), \\
&\tilde{\omega}_3 = -\omega_3 = \tfrac{1}{2}(\partial_r f_1 + \partial_r f_2), \\
&E = \tfrac{1}{2}(\partial_r^2 f_1 + \partial_r^2 f_2) - \tfrac{1}{4}(\partial_r f_1 + \partial_r f_2)^2 + \tfrac{1}{2}(\partial_r f_1 + \partial_r f_2)^2 \,.
\end{aligned} \tag{2.3.p}$$

Consequently,

$$\begin{aligned}
&\langle E\phi, \rho\rangle = \tfrac{1}{2}(\varrho_1 + \varrho_2) + \tfrac{1}{4}(\varepsilon_1 + \varepsilon_2)^2, \\
&\tfrac{2}{3}(\langle \phi_{;mm}, \rho\rangle + \langle \phi, \rho_{;mm}\rangle) = \tfrac{2}{3}(\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)^2 - \varrho_1 - \varrho_2), \\
&-\tfrac{2}{3}L_{aa}\langle \phi, \rho\rangle_{;m} = -\tfrac{2}{3}(\varepsilon_1 + \varepsilon_2)^2, \\
&c_{10}R_{amma}\langle \phi, \rho\rangle = -c_{10}(\varepsilon_1^2 + \varepsilon_2^2 + \varrho_1 + \varrho_2), \\
&c_{11}L_{aa}L_{bb}\langle \phi, \rho\rangle = c_{11}(\varepsilon_1 + \varepsilon_2)^2, \\
&c_{12}L_{ab}L_{ab}\langle \phi, \rho\rangle = c_{12}(\varepsilon_1^2 + \varepsilon_2^2).
\end{aligned}$$

Setting $\beta_3 = 0$ then yields the three equations from which Assertion (5) follows

$$\begin{array}{lll}
0 = \{\tfrac{1}{2} - \tfrac{2}{3} - c_{10}\}(\varrho_1 + \varrho_2) & \text{so} & c_{10} = -\tfrac{1}{6}, \\
0 = \{\tfrac{1}{4} + \tfrac{1}{3} - \tfrac{2}{3} + c_{11}\}(\varepsilon_1 + \varepsilon_2)^2 & \text{so} & c_{11} = \tfrac{1}{12}, \\
0 = \{-c_{10} + c_{12}\}(\varepsilon_1^2 + \varepsilon_2^2), & \text{so} & c_{12} = -\tfrac{1}{6}. \quad \square
\end{array}$$

The proof of the final assertion of Theorem 2.3.3 follows much the same lines so we shall omit the proof and refer instead to [45]. We shall mention only one important feature. Instead of choosing the interior integrand to be $-\frac{1}{2}\langle D^2\phi, \rho\rangle$, we have chosen the interior integrand to be $-\frac{1}{2}\langle D\phi, \tilde{D}\rho\rangle$. This introduces additional boundary integrands we simply absorb in the universal constants. Since this integrand is symmetric, by Lemma 2.1.3, the boundary integrands are symmetric in ϕ and ρ as well which vastly simplifies the invariance theory involved.

2.4 Heat content asymptotics for Robin boundary conditions

Throughout this section, we shall let D be an operator of Laplace type on $C^\infty(V)$. If S is an auxiliary endomorphism of $V|_{\partial M}$, then the *Robin boundary operator* is given by

$$\mathcal{B}\phi := (\phi_{;m} + S\phi)|_{\partial M} \,.$$

By Lemma 1.5.2, the adjoint boundary operator is again of this type since

$$\tilde{\mathcal{B}}\rho := (\rho_{;m} + \tilde{S}\rho)|_{\partial M} \,.$$

Both $(D, \mathcal{B})$ and $(\tilde{D}, \tilde{\mathcal{B}})$ are admissible. The following result [45, 132] is the main result of this section.

Theorem 2.4.1 *Let D be an operator of Laplace type on a compact Riemannian manifold with smooth boundary. Impose Robin boundary conditions.*

1. $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle dx$.

2. $\beta_1(\phi,\rho,D,\mathcal{B}) = 0$.

3. $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M \langle D\phi,\rho\rangle dx + \int_{\partial M}\langle\mathcal{B}\phi,\rho\rangle dy$.

4. $\beta_3(\phi,\rho,D,\mathcal{B}) = \frac{2}{3}\cdot\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle dy$.

5. $\beta_4(\phi,\rho,D,\mathcal{B}) = \frac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle dx$
$+\int_{\partial M}\{-\frac{1}{2}\langle\mathcal{B}\phi,\tilde{D}\rho\rangle - \frac{1}{2}\langle D\phi,\tilde{\mathcal{B}}\rho\rangle + \langle(\frac{1}{2}S+\frac{1}{4}L_{aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle\}dy$.

6. $\beta_5(\phi,\rho,D,\mathcal{B}) = \frac{2}{\sqrt{\pi}}\int_{\partial M}\{-\frac{4}{15}(\langle\mathcal{B}D\phi,\tilde{\mathcal{B}}\rho\rangle + \langle\mathcal{B}\phi,\tilde{\mathcal{B}}\tilde{D}\rho\rangle)$
$-\frac{2}{15}\langle(\mathcal{B}\phi)_{:a},(\tilde{\mathcal{B}}\rho)_{:a}\rangle + \langle(\frac{2}{15}E+\frac{4}{15}S^2+\frac{4}{15}SL_{aa}+\frac{1}{30}L_{aa}L_{bb}$
$+\frac{1}{15}L_{ab}L_{ab}-\frac{1}{15}R_{amam})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle\}dy$.

7. $\beta_6(\phi,\rho,D,\mathcal{B}) = -\frac{1}{6}\int_M\langle D^2\phi,\tilde{D}\rho\rangle dx + \int_{\partial M}\{\frac{1}{6}\langle\mathcal{B}D\phi,\tilde{D}\rho\rangle + \frac{1}{6}\langle D^2\phi,\tilde{\mathcal{B}}\rho\rangle$
$+\frac{1}{6}\langle\mathcal{B}\phi,\tilde{D}^2\rho\rangle - \frac{1}{6}\langle S\mathcal{B}D\phi,\tilde{\mathcal{B}}\rho\rangle - \frac{1}{6}\langle S\mathcal{B}\phi,\tilde{\mathcal{B}}\tilde{D}\rho\rangle - \frac{1}{12}\langle L_{aa}\mathcal{B}D\phi,\tilde{\mathcal{B}}\rho\rangle$
$-\frac{1}{12}\langle L_{aa}\mathcal{B}\phi,\tilde{\mathcal{B}}\tilde{D}\rho\rangle + \langle(\frac{1}{24}E_{;m}+\frac{1}{12}EL_{aa}+\frac{1}{48}L_{ab}L_{ab}L_{cc}$
$+\frac{1}{24}L_{ab}L_{ac}L_{bc}-\frac{1}{48}R_{ambm}L_{ab}+\frac{1}{48}R_{abcb}L_{ac}-\frac{1}{24}R_{amam}L_{bb}$
$+\frac{1}{96}\tau_{;m}+\frac{1}{48}L_{ab:ab}+\frac{1}{12}SL_{aa}L_{bb}+\frac{1}{12}SL_{ab}L_{ab}-\frac{1}{12}SR_{amam}$
$+\frac{1}{12}(SE+ES)+\frac{1}{4}S^2L_{aa}+\frac{1}{6}S^3+\frac{1}{6}S_{:aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle - \frac{1}{12}L_{aa}\langle(\mathcal{B}\phi)_{:b},(\tilde{\mathcal{B}}\rho)_{:b}\rangle$
$-\frac{1}{12}L_{ab}\langle(\mathcal{B}\phi)_{:a},(\tilde{\mathcal{B}}\rho)_{:b}\rangle - \frac{1}{6}\langle S(\mathcal{B}\phi)_{:a},(\tilde{\mathcal{B}}\rho)_{:a}\rangle - \frac{1}{24}\langle\Omega_{am}(\mathcal{B}\phi)_{:a},\tilde{\mathcal{B}}\rho\rangle$
$+\frac{1}{24}\langle\Omega_{am}\mathcal{B}\phi,(\tilde{\mathcal{B}}\rho)_{:a}\rangle\}dy$.

Assertion (1) of Theorem 2.4.1 follows from Lemma 2.1.1. The rest of this section is devoted to the proof Assertions (2), (3), (4), and (5).

2.4.1 *Proof of Theorem 2.4.1 (2)*

By Lemma 2.2.13 there is a universal constant c_0 so that

$$\beta_1(\phi,\rho,D,\mathcal{B}) = c_0\int_{\partial M}\langle\phi,\rho\rangle dy\,. \tag{2.4.a}$$

Lemma 2.1.4 is a very powerful tool in this setting. If $\mathcal{B}\phi = 0$, then

$$\beta_1(\phi,\rho,D,\mathcal{B}) = -2\beta_{-1}(D\phi,\rho,D,\mathcal{B}) = 0\,. \tag{2.4.b}$$

Since by Lemma 1.4.1 we can choose ϕ so $\mathcal{B}\phi = 0$ and so $\phi|_{\partial M}$ is arbitrary. We use Equations (2.4.a) and (2.4.b) to establish Assertion (2) of Theorem 2.4.1 by showing $c_0 = 0$. $\square$

2.4.2 Proof of Theorem 2.4.1 (3)

There exist universal constants so

$$\begin{aligned}\beta_2(\phi,\rho,D,\mathcal{B}) &= -\int_M \langle D\phi,\rho\rangle dx \\ &\quad + \int_{\partial M}\Big\{c_1\langle\mathcal{B}\phi,\rho\rangle + c_2\langle\phi,\tilde{\mathcal{B}}\rho\rangle + c_3\langle S\phi,\rho\rangle + c_4 L_{aa}\langle\phi,\rho\rangle\Big\}dy\,.\end{aligned} \tag{2.4.c}$$

If $\mathcal{B}\phi = 0$, then $\beta_2(\phi,\rho,D,\mathcal{B}) = -\beta_0(D\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle dx$ by Lemma 2.1.4. Consequently,

$$0 = \int_{\partial M}\Big\{c_2\langle\phi,\tilde{\mathcal{B}}\rho\rangle + c_3\langle S\phi,\rho\rangle + c_4 L_{aa}\langle\phi,\rho\rangle\Big\}dy\,.$$

It now follows that $c_2 = c_3 = c_4 = 0$. Furthermore, by Lemma 2.1.3,

$$\begin{aligned} 0 &= \beta_2(\rho,\phi,\tilde D,\tilde{\mathcal{B}}) - \beta_2(\phi,\rho,D,\mathcal{B}) \\ &= \int_M\Big\{\langle D\phi,\rho\rangle - \langle\phi,\tilde D\rho\rangle\Big\}dx - c_1\int_{\partial M}\Big\{\langle\phi_{;m},\rho\rangle - \langle\phi,\rho_{;m}\rangle\Big\}dy\,.\end{aligned}$$

Thus by the Green's formula, which is given in Lemma 1.4.17, $c_1 = 1$. □

2.4.3 Proof of Theorem 2.4.1 (4)

Take into account the symmetry of Lemma 2.1.3. By Lemma 2.2.13, there exist universal constants so

$$\begin{aligned}\beta_3(\phi,\rho,D,\mathcal{B}) = \int_{\partial M}\Big\{&c_5\langle\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle + c_6(\langle\phi_{;mm},\rho\rangle + \langle\phi,\rho_{;mm}\rangle) \\ &+c_7 L_{aa}\langle\phi,\rho\rangle_{;m} + c_8(\langle S\mathcal{B}\phi,\rho\rangle + \langle\phi,\tilde S\tilde{\mathcal{B}}\rho\rangle) + c_9\langle\phi_{:a},\rho_{:a}\rangle \\ &+\langle(c_{10}S^2 + c_{11}SL_{aa} + c_{12}\tau + c_{13}R_{amma} + c_{14}L_{aa}L_{bb} \\ &+c_{15}L_{ab}L_{ab})\phi,\rho\rangle\Big\}dy\,.\end{aligned}$$

If $\mathcal{B}\phi = 0$, then $\beta_3(\phi,\rho,D,\mathcal{B}) = 0$ by Lemma 2.1.4 since $\beta_1 = 0$. This shows

$$c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{12} = c_{13} = c_{14} = c_{15} = 0\,.$$

We adopt the notation of Lemma 2.1.15 to determine c_5. Let

$$\begin{array}{ll} M := [0,1], & b\in C^\infty(M), \\ A := \partial_x + b, & A^* := -\partial_x + b, \\ D_1 := A^*A, & D_2 := AA^*, \\ \mathcal{B}_1\phi := A\phi|_{\partial M}, & \mathcal{B}_2\phi := \phi|_{\partial M}\,. \end{array} \tag{2.4.d}$$

One can apply Lemma 2.1.15 and Theorem 2.3.3 to see that

$$\beta_3(\phi,\rho,D_1,\mathcal{B}_1) = \tfrac{2}{\sqrt{\pi}}c_5\int_{\partial M}\langle A\phi,A\rho\rangle dy$$

$$= \quad -\tfrac{2}{3}\beta_1(A\phi, A\rho, D_2, \mathcal{B}_2) = \tfrac{2}{3}\tfrac{2}{\sqrt{\pi}}\int_{\partial M}\langle A\phi, A\rho\rangle dy\,.$$

It now follows that $c_5 = \frac{2}{3}$. $\square$

2.4.4 Proof of Theorem 2.4.1 (5)

To preserve the symmetry of Lemma 2.1.3, we express

$$\beta_4(\phi,\rho,D,\mathcal{B}) = \tfrac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle + \int_{\partial M}\beta_4^{\partial M}(\phi,\rho,D,\mathcal{B})dy\,.$$

We then have $\beta_4^{\partial M}$ is symmetric in the roles of ϕ and ρ. If $\mathcal{B}\phi = 0$, then Lemma 2.1.3, Lemma 2.1.4, and Theorem 2.4.1 imply

$$\begin{aligned}
&\beta_4(\phi,\rho,D,\mathcal{B}) = -\tfrac{1}{2}\beta_2(D\phi,\rho,D,\mathcal{B}) = -\tfrac{1}{2}\beta_2(\rho,D\phi,\tilde{D},\tilde{\mathcal{B}})\\
&\quad = \tfrac{1}{2}\int_{\partial M}\langle\tilde{D}\rho, D\phi\rangle dx - \tfrac{1}{2}\int_{\partial M}\langle\tilde{\mathcal{B}}\rho, D\phi\rangle dy \text{ so}\\
&\int_{\partial M}\beta_4^{\partial M}(\rho,\phi,D,\mathcal{B}) = -\tfrac{1}{2}\int_{\partial M}\langle D\phi,\tilde{\mathcal{B}}\rho\rangle dy\,.
\end{aligned}$$

As previously, this identity shows that many of the terms which might appear in $\beta_4^{\partial M}$ have coefficient 0. We conclude there exist universal constants so that

$$\beta_4(\phi,\rho,D,\mathcal{B}) = -\tfrac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle dx \tag{2.4.e}$$

$$+\textstyle\int_{\partial M}\Big\{-\tfrac{1}{2}(\langle\mathcal{B}\phi,\tilde{D}\rho\rangle + \langle D\phi,\tilde{\mathcal{B}}\rho\rangle) \tag{2.4.f}$$

$$+\langle(c_{16}S + c_{17}L_{aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle\Big\}dy\,.$$

Lemma 2.4.2

1. $c_{16} = \frac{1}{2}$.

2. $c_{17} = \frac{1}{4}$.

Proof: Adopt the notation of Equation (2.4.d) and set $n = 4$ in Lemma 2.1.15. We suppose ρ vanishes identically near $r = 1$ so only the boundary component $r = 0$ where ∂_x is the inward unit normal is relevant. We use Theorem 2.3.3 (3) and integrate by parts to see

$$\begin{aligned}
&\beta_4(\phi,\rho,D_1,\mathcal{B}_1) = -\tfrac{1}{2}\beta_2(A\phi,A\rho,D_2,\mathcal{B}_2)\\
= \quad &\tfrac{1}{2}\int_M\langle(AA^*)A\phi,A\rho\rangle dx + \tfrac{1}{2}\int_{\partial M}\langle A\phi,(A\rho)_{;m}\rangle dy\\
= \quad &\tfrac{1}{2}\int_M\Big\{\langle D_1\phi,\tilde{D}_1\rho\rangle\} + \partial_x\langle D_1\phi,A\rho\rangle\Big\}dx + \tfrac{1}{2}\int_{\partial M}\langle A\phi,\partial_x(A\rho)\rangle dy\\
= \quad &\tfrac{1}{2}\int_M\langle D_1\phi,\tilde{D}_1\rho\rangle dx + \int_{\partial M}\Big\{-\tfrac{1}{2}\langle D_1\phi,A\rho\rangle + \tfrac{1}{2}\langle A\phi,\partial_x(A\rho)\rangle\Big\}dy
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\int_M \langle D_1\phi, \tilde{D}_1\rho\rangle dx \\
&+ \int_{\partial M}\Big\{-\tfrac{1}{2}\langle D_1\phi, A\rho\rangle - \tfrac{1}{2}\langle A\phi, (-\partial_x + b)A\rho\rangle + \tfrac{1}{2}\langle A\phi, bA\rho\rangle\Big\}dy \\
&= \tfrac{1}{2}\int_M \langle D_1\phi, \tilde{D}_1\rho\rangle dx \\
&+ \int_{\partial M}\Big\{-\tfrac{1}{2}\langle D_1\phi, \tilde{\mathcal{B}}_1\rho\rangle - \tfrac{1}{2}\langle \mathcal{B}_1\phi, \tilde{D}_1\rho\rangle + \tfrac{1}{2}\langle S\mathcal{B}\phi, \tilde{\mathcal{B}}_1\rho\rangle\Big\}dy\,.
\end{aligned}
$$

Comparing this with the expression given in Equation (2.4.e) establishes the first assertion by showing

$$c_{16} = \tfrac{1}{2}\,.$$

We use Lemma 2.1.9 to complete the proof. We adopt the notation of Equation (2.3.n) and take a warped product metric

$$ds_M^2 := dr^2 + e^{2f_1(r)}d\theta_1^2 + e^{2f_2(r)}d\theta_2^2$$

on the cylinder $M := S^1 \times S^1 \times [0,1]$ where, for $a = 1, 2$, one has

$$
\begin{aligned}
&f_a(0) = 0, \quad \partial_r f_a(0) = \quad \varepsilon_a, \quad \partial_r^2 f_a(0) = \varrho_a, \\
&f_a(1) = 0, \quad \partial_r f_a(1) = -\varepsilon_a, \quad \partial_r^2 f_a(1) = \varrho_a\,.
\end{aligned}
$$

We set

$$
\begin{aligned}
&D_1 := -\partial_r^2, && \mathcal{B}_1 := \varepsilon\partial_r, \\
&D := -\partial_r^2 - e^{-2f_1}\partial_1^\theta - e^{-2f_2}\partial_{\theta_1}^2, && \mathcal{B} := \varepsilon\partial_r
\end{aligned}
$$

where $\varepsilon(0) = +1$ and $\varepsilon(1) = -1$ to ensure that $\varepsilon\partial_r$ is the inward unit normal. By Equations (2.3.o) and (2.3.p), we have

$$
\begin{aligned}
&L_{aa} = -\varepsilon_1 - \varepsilon_2, && \omega_r^D = -\tfrac{1}{2}(\partial_r f_1 + \partial_r f_2), \\
&S_1 = 0, && S = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)\,.
\end{aligned}
$$

We apply Lemma 2.1.9 twice with $\phi = 1$ and $\rho = e^{-f_1 - f_2}$ to see

$$\beta_4(\phi, \rho, D, \mathcal{B}) = (2\pi)^2\beta_4(1, 1, -\partial_r^2, \mathcal{B}_1) = 0\,.$$

On the other hand, we can use Equation (2.4.e) and the computations performed above to see that

$$\beta_4(\phi, \rho, D, \mathcal{B}) = \int_{\partial M}\Big\{\frac{1}{2}c_{16}(\varepsilon_1 + \varepsilon_2) + c_{17}(-\varepsilon_1 - \varepsilon_2)\Big\}dy\,.$$

Thus $0 = \frac{1}{2}c_{16} - c_{17}$ so $c_{17} = \frac{1}{2}c_{16} = \frac{1}{4}$. This completes the proof of the Lemma and thereby also completes the proof of Theorem 2.4.1 (5). □

The proof of the remaining two assertions of Theorem 2.4.1 is similar and is omitted in the interest of brevity; we refer to [45, 132] for further details. Combinatorially speaking, the computation of β_n for Neumann boundary conditions entails roughly the same amount of work as the computation of β_{n-2} for Dirichlet boundary conditions.

2.5 Heat content asymptotics for mixed boundary conditions

Let D be an operator of Laplace type on a smooth vector bundle V. We adopt the notation of Section 1.5.3. Let χ be an auxiliary endomorphism of V which is defined near ∂M so that $\chi^2 = \mathrm{Id}$ and so that $\chi_{;m} = 0$. Let $\Pi_\pm := \frac{1}{2}(\mathrm{Id} \pm \chi)$ and $\tilde\Pi_\pm := \frac{1}{2}(\mathrm{Id} \pm \tilde\chi)$ be the associated projections on the ± 1 eigenspaces $V_\pm$ and $V_\pm^*$ of χ and of $\tilde\chi$, respectively. Let S be an auxiliary endomorphism of V_+ which we extend to be zero on V_-. The mixed boundary operators are then defined by

$$\begin{aligned} \mathcal{B}\phi &:= \Pi_+(\phi_{;m} + S\phi)|_{\partial M} \oplus \Pi_-\phi|_{\partial M} \\ \tilde{\mathcal{B}}\rho &:= \tilde\Pi_+(\rho_{;m} + \tilde S\rho)|_{\partial M} \oplus \tilde\Pi_-\rho|_{\partial M}\,. \end{aligned} \tag{2.5.a}$$

Then $\tilde{\mathcal{B}}$ defines the adjoint boundary condition; both $(D,\mathcal{B})$ and $(\tilde D,\tilde{\mathcal{B}})$ are admissible. Let

$$\phi_\pm := \Pi_\pm\phi \quad \text{and} \quad \rho_\pm := \Pi_\pm\rho\,.$$

Since $\chi_{;m} = 0$,

$$\phi_{\pm;m} = \Pi_\pm(\phi_{;m}) \quad \text{and} \quad \rho_{\pm;m} = \tilde\Pi_\pm(\phi_{;m})\,.$$

Since the tangential covariant derivatives of χ need not vanish, we do not necessarily have equality between $\phi_{\pm:a}$ and $\Pi_\pm(\phi_{:a})$ nor do we necessarily have equality between $\rho_{\pm:a}$ and $\tilde\Pi_\pm(\rho_{:a})$.

The following is the main result [132] of this section:

Theorem 2.5.1 *Let D be an operator of Laplace type on a compact Riemannian manifold with smooth boundary ∂M. Let $\mathcal{B}$ be the mixed boundary operator of Equation* (2.5.a)*. Then*

1. $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle dx$.
2. $\beta_1(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\phi_-,\rho_-\rangle dy$.
3. $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\{\langle\phi_{+;m} + S\phi_+,\rho_+\rangle + \langle\frac{1}{2}L_{aa}\phi_-,\rho_-\rangle$
$-\langle\phi_-,\rho_{-;m}\rangle\}dy$.
4. $\beta_3(\phi,\rho,D,\mathcal{B}) = \frac{2}{\sqrt{\pi}}\int_{\partial M}\{-\frac{2}{3}\langle\phi_{-;mm},\rho_-\rangle - \frac{2}{3}\langle\phi_-,\rho_{-;mm}\rangle$
$+\frac{2}{3}L_{aa}\langle\phi_-,\rho_-\rangle_{;m} + \langle(-\frac{1}{12}L_{aa}L_{bb} + \frac{1}{6}L_{ab}L_{ab} + \frac{1}{6}R_{amma})\phi_-,\rho_-\rangle$
$+\frac{2}{3}\langle\phi_{+;m} + S\phi_+,\rho_{+;m} + \tilde S\rho_+\rangle - \langle E\phi_-,\rho_-\rangle + \langle\phi_{-:a},\rho_{-:a}\rangle$
$+\frac{2}{3}\langle\phi_{+:a},\rho_{-:a}\rangle + \frac{2}{3}\langle\phi_{-:a},\rho_{+:a}\rangle - \frac{2}{3}\langle E\phi_-,\rho_+\rangle - \frac{2}{3}\langle E\phi_+,\rho_-\rangle\}dy$.

We shall need the following result in Section 2.11 to compute the heat content asymptotics of non-minimal operators. Let $\phi = \phi_i e_i$ and let $\rho = \rho_i e_i$ be cotangent vector fields. The normal components are then ϕ_m and ρ_m while the tangential components are $\phi_a e_a$ and $\rho_a e_a$. Let $(\cdot,\cdot)$ denote the natural extension of the metric to an innerproduct on the exterior algebra $\Lambda(M)$.

Theorem 2.5.2 *Let $\Delta^1 := \delta d + d\delta$ be the Laplacian on $C^\infty(\Lambda^1(M))$.*

1. *Let $\mathcal{B}$ define absolute boundary conditions. Then*
   (a) $\beta_0(\phi,\rho,\Delta^1,\mathcal{B}) = \int_M (\phi,\rho)dx$.
   (b) $\beta_1(\phi,\rho,\Delta^1,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\phi_m\rho_m dy$.
   (c) $\beta_2(\phi,\rho,\Delta^1,\mathcal{B}) = -\int_M\{(\delta\phi,\delta\rho)+(d\phi,d\rho)\}dx + \int_{\partial M}\{-\phi_{a:a}\rho_m$ $-\phi_m\rho_{a:a} - \phi_{m;m}\rho_m - \phi_m\rho_{m;m} + \frac{3}{2}L_{aa}\phi_m\rho_m\}dy$.
2. *Let $\mathcal{B}$ define relative boundary conditions. Then*
   (a) $\beta_0(\phi,\rho,\Delta^1,\mathcal{B}) = \int_M (\phi,\rho)dx$.
   (b) $\beta_1(\phi,\rho,\Delta^1,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\phi_a\rho_a dy$.
   (c) $\beta_2(\phi,\rho,\Delta^1,\mathcal{B}) = -\int_M\{(\delta\phi,\delta\rho)+(d\phi,d\rho)\}dx + \int_{\partial M}\{-\phi_{a:a}\rho_m$ $-\phi_m\rho_{a:a} - \phi_{a;m}\rho_a - \phi_a\rho_{a;m} + L_{ab}\phi_a\rho_b + \frac{1}{2}L_{aa}\phi_b\rho_b\}dy$.

2.5.1 The proof of Theorem 2.5.1 (1-3)

If $V = V_+ \oplus V_-$ and $D = D_+ \oplus D_-$, then the two boundary conditions do not interact so Lemma 2.1.7 implies the heat content asymptotics decouple. The failure of such a splitting to hold true in general is measured by the failure of χ to commute with ∇ and with E. Invariants measuring $[E,\chi]$ have order at least 2; invariants measuring $[\chi,\nabla]$ must have at least one tangential index to contract. Since all indices are contracted in pairs, such invariants also must have order at least 2. Thus these invariants do not arise in β_0, β_1, and β_2 so Assertions (1), (2), and (3) of Theorem 2.5.1 follow by Lemma 2.1.7 from the corresponding assertions of Theorems 2.3.3 and 2.4.1. $\square$

2.5.2 The proof of Theorem 2.5.1 (4)

As $\chi^2 = \mathrm{Id}$,

$$0 = \chi_{:a}\chi + \chi\chi_{:a} \quad \text{and} \quad 0 = \chi\chi_{:aa} + 2\chi_{:a}\chi_{:a} + \chi_{:aa}\chi\,.$$

Keeping in mind the symmetry property given in Lemma 2.1.3, we see there are universal constants so

$$\begin{aligned}\beta_3(\phi,\rho,D,\mathcal{B}) = \tfrac{2}{\sqrt{\pi}}\int_{\partial M}\Big\{ &-\tfrac{2}{3}\langle\phi_{-;mm},\rho_-\rangle - \tfrac{2}{3}\langle\phi_-,\rho_{-;mm}\rangle\\ &+\tfrac{2}{3}L_{aa}\langle\phi_-,\rho_-\rangle_{;m} + \langle(-\tfrac{1}{12}L_{aa}L_{bb}+\tfrac{1}{6}L_{ab}L_{ab}+\tfrac{1}{6}R_{amma})\phi_-,\rho_-\rangle\\ &+\tfrac{2}{3}\langle\phi_{+;m}+S\phi_+,\rho_{+;m}+\tilde{S}\rho_+\rangle - \langle E\phi_-,\rho_-\rangle + \langle\phi_{-:a},\rho_{-:a}\rangle\\ &+c_0(\langle E\phi_+,\rho_-\rangle + \langle E\phi_-,\rho_+\rangle) + c_1(\langle\phi_{+:a},\rho_{-:a}\rangle + \langle\phi_{-:a},\rho_{+:a}\rangle)\\ &+c_2\langle\chi_{:aa}\phi,\rho\rangle + c_3\langle\chi_{:a}\chi_{:a}\phi,\rho\rangle + c_4\langle\chi_{:a}\phi_{:a},\rho\rangle\\ &+c_5\langle\chi\chi_{:aa}\phi,\rho\rangle + c_6\langle\chi\chi_{:a}\chi_{:a}\phi,\rho\rangle + c_7\langle\chi\chi_{:a}\phi_{:a},\rho\rangle\Big\}dy\,. \end{aligned} \qquad (2.5.b)$$

Lemma 2.5.3

1. $c_0 = -\frac{2}{3}$.

2. $c_1 = \frac{2}{3}$, *and* $c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$.

Proof: Let $M = [0,1]$ and let $V = M \times \mathbb{R}^2$. Take

$$E := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \chi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\phi := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \rho := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $D := -\partial_x^2 - E$ and let $S = 0$. By Equation (2.5.b),

$$\beta_3(\phi, \rho, D, \mathcal{B}) = \tfrac{4}{\sqrt{\pi}} c_0 \,. \tag{2.5.c}$$

We show that $\mathcal{B}\phi = 0$ by computing

$$\mathcal{B}\phi = \Pi_+ \phi_{;m} \oplus \Pi_- \phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Lemma 2.1.4 now implies that

$$\begin{aligned} & \beta_3(\phi, \rho, D, \mathcal{B}) = -\tfrac{2}{3}\beta_1(D\phi, \rho, D, \mathcal{B}) & (2.5.d) \\ = \quad & \tfrac{4}{3\sqrt{\pi}} \textstyle\int_{\partial M} \langle \Pi_- D\phi, \rho \rangle dy = -\tfrac{8}{3\sqrt{\pi}} \,. \end{aligned}$$

Equations (2.5.c) and (2.5.d) imply $c_0 = -\frac{2}{3}$, which proves Assertion (1).

To prove the second assertion, we give an argument similar to that used in the proof of Lemma 2.3.7. Let $M := S^1 \times [0,1]$ be the cylinder and let $V := M \times \mathbb{R}^k$ be a trivial vector bundle of rank k. Let $\varepsilon \in M_n(\mathbb{R})$ be a constant matrix. We let

$$D := -\partial_r^2 - \partial_\theta^2 - 2\varepsilon\partial_\theta \,.$$

We take ϕ and ρ to be constant vectors. Set $S = 0$. Let $\chi \in M_k(\mathbb{R})$ be a constant matrix with $\chi^2 = \mathrm{Id}$. Since

$$(-\partial_\theta^2 - 2\varepsilon\partial_\theta)\phi = 0 \,,$$

Lemma 2.1.9 shows that

$$\beta_3(\phi, \rho, D, \mathcal{B}) = 2\pi\beta_3(\phi, \rho, -\partial_r^2, \mathcal{B}) \,. \tag{2.5.e}$$

The relevant structures are given in Equation (2.3.l)

$$\nabla_{\partial_r}\phi = 0, \quad \nabla_{\partial_r}\rho = 0, \quad \nabla_{\partial_\theta}\phi = \varepsilon\phi, \quad \nabla_{\partial_\theta}\rho = -\tilde{\varepsilon}\rho, \quad E = -\varepsilon^2 \,.$$

Let $\{e_1 := \partial_\theta, e_2 := \partial_r\}$ be the coordinate frame. Then we have

$$\chi_{:1} = [\varepsilon, \chi] \quad \text{and} \quad \chi_{:11} = [\varepsilon, [\varepsilon, \chi]] \,.$$

The right hand side of Equation (2.5.e) is independent of ε. Because the contributions made by the invariants $-\langle E\phi_-, \rho_- \rangle$ and $\langle \phi_{-:a}, \rho_{-:a} \rangle$ cancel,

$$0 = \int_{\partial M} \Big\{ -(c_1 + c_0)(\langle \varepsilon^2\phi_+, \rho_- \rangle + \langle \varepsilon^2\phi_-, \rho_+ \rangle)$$

$$
+c_2\langle[\varepsilon,[\varepsilon,\chi]]\phi,\rho\rangle + c_3\langle[\varepsilon,\chi]^2\phi,\rho\rangle + c_4\langle[\varepsilon,\chi]\varepsilon\phi,\rho\rangle
$$
$$
\left.+c_5\langle\chi[\varepsilon,[\varepsilon,\chi]]\phi,\rho\rangle + c_6\langle\chi[\varepsilon,\chi]^2\phi,\rho\rangle + c_7\langle\chi[\varepsilon,\chi]\varepsilon\phi,\rho\rangle\right\}dy\,.
$$

This identity for **every** χ,ε implies that the relations of Assertion (2) hold. This completes the proof of the Lemma and thereby the proof of Theorem 2.5.1 as well. □

2.5.3 The proof of Theorem 2.5.2

Assertions (1a) and (2a) are immediate. By Lemma 1.5.4, one has with absolute boundary conditions $\mathcal{B}_a$ that

$$
\begin{aligned}
&\phi_+ = \phi_a e_a, \quad \rho_+ = \rho_a e_a, \quad \phi_- = \phi_m e_m, \quad \rho_- = \rho_m e_m, \\
&S\phi = -L_{ab}\phi_b e_b
\end{aligned}
\tag{2.5.f}
$$

while relative boundary conditions $\mathcal{B}_r$ one has that

$$
\begin{aligned}
&\phi_+ = \phi_m e_m, \quad \rho_+ = \rho_m e_m, \quad \phi_- = \phi_a e_a, \quad \rho_- = \rho_a e_a, \\
&S\phi = -L_{aa}\phi_m e_m\,.
\end{aligned}
\tag{2.5.g}
$$

Assertions (1b) and (2b) now follow from Theorem 2.5.1 (2).

To establish Assertions (1c) and (2c), we must rewrite the interior boundary integrand. Let $\mathfrak{i}(e_m)$ be interior multiplication by e_m. By Lemma 1.4.16,

$$
\int_M \{(d\phi,\rho) - (\phi,\delta\rho)\}dx = -\int_{\partial M}(\phi,\mathfrak{i}(e_m)\rho)dy\,.
$$

Therefore, one has that

$$
\begin{aligned}
& -\int_M(\Delta^1\phi,\rho)dx = -\int_M\{(\delta d\phi,\rho)+(d\delta\phi,\rho)\}dx \\
= \quad & -\int_M\{(d\phi,d\rho)+(\delta\phi,\delta\rho)\}dx \\
& +\int_{\partial M}\{-(\mathfrak{i}(e_m)d\phi,\rho)+(\delta\phi,\mathfrak{i}(e_m)\rho)\}dy \\
= \quad & -\int_M\Big\{(d\phi,d\rho)+(\delta\phi,\delta\rho)\Big\}dx \\
& +\int_{\partial M}\Big\{-\phi_{a;m}\rho_a+\phi_{m;a}\rho_a-\phi_{i;i}\rho_m\Big\}dy\,.
\end{aligned}
$$

We use the relations $\phi_{b;a} = \phi_{b:a} - L_{ab}\phi_m$ and $\phi_{m;a} = \phi_{m:a} + L_{ab}\phi_b$ and integrate by parts once again to compute

$$
\begin{aligned}
& -\int_M(\Delta^1\phi,\rho)dx \\
= \quad & -\int_M\Big\{(d\phi,d\rho)+(\delta\phi,\delta\rho)\Big\}dx + \int_{\partial M}\Big\{-\phi_{a;m}\rho_a
\end{aligned}
\tag{2.5.h}
$$

$$
\begin{aligned}
&\quad +\phi_{m:a}\rho_a - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m + L_{aa}\phi_m\rho_m + L_{ab}\phi_a\rho_b\Big\}dy\\
&= -\int_M \{(d\phi, d\rho) + (\delta\phi, \delta\rho)\Big\}dx + \int_{\partial M}\Big\{ -\phi_{a;m}\rho_a\\
&\quad -\phi_m\rho_{a:a} - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m + L_{aa}\phi_m\rho_m + L_{ab}\phi_a\rho_b\Big\}dy\,.
\end{aligned}
$$

Assertion (1c) follows from Theorem 2.5.1 (3) and Equation (2.5.f) as

$$
\begin{aligned}
&\beta_2(\phi, \rho, \Delta^1, \mathcal{B}_a) = -\int_M (\Delta^1\phi, \rho)dx\\
+&\int_{\partial M}\Big\{(\phi_{+;m} + S\phi_+, \rho_+) + \tfrac{1}{2}L_{aa}(\phi_-, \rho_-) - (\phi_-, \rho_{-;m})\Big\}dy\\
=& -\int_M \Big\{(\delta\phi, \delta\rho) + (d\phi, d\rho)\Big\}dx\\
+&\int_{\partial M}\Big\{ -\phi_{a;m}\rho_a - \phi_m\rho_{a:a} - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m + L_{aa}\phi_m\rho_m\\
+&\ L_{ab}\phi_a\rho_b + \tfrac{1}{2}L_{aa}(\phi_-, \rho_-) + (\phi_{+;m} + S\phi_+, \rho_+) - (\phi_-, \rho_{-;m})\Big\}dy\\
=& -\int_M \Big\{(\delta\phi, \delta\rho) + (d\phi, d\rho)\Big\}dx\\
+&\int_{\partial M}\Big\{ -\phi_m\rho_{a:a} - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m - \phi_m\rho_{m;m} + \tfrac{3}{2}L_{aa}\phi_m\rho_m\Big\}dy\,.
\end{aligned}
$$

Similarly, we use Equation (2.5.g) to establish Assertion (2c) by checking

$$
\begin{aligned}
&\beta_2(\phi, \rho, \Delta^1, \mathcal{B}_r) = -\int_M (\Delta^1\phi, \rho)dx\\
+&\int_{\partial M}\Big\{(\phi_{+;m} + S\phi_+, \rho_+) + \tfrac{1}{2}L_{aa}(\phi_-, \rho_-) - (\phi_-, \rho_{-;m})\Big\}dy\\
=& -\int_M \Big\{(\delta\phi, \delta\rho) + (d\phi, d\rho)\Big\}dx\\
+&\int_{\partial M}\Big\{ -\phi_{a;m}\rho_a - \phi_m\rho_{a:a} - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m + L_{aa}\phi_m\rho_m\\
+&\ L_{ab}\phi_a\rho_b + \tfrac{1}{2}L_{aa}(\phi_-, \rho_-) + (\phi_{+;m} + S\phi_+, \rho_+) - (\phi_-, \rho_{-;m})\Big\}dy\\
=& -\int_M \Big\{(\delta\phi, \delta\rho) + (d\phi, d\rho)\Big\}dx\\
+&\int_{\partial M}\Big\{ -\phi_m\rho_{a:a} - \phi_{a:a}\rho_m - \phi_{a;m}\rho_a - \phi_a\rho_{a;m} + \tfrac{1}{2}L_{aa}\phi_b\rho_b\\
&+L_{ab}\phi_a\rho_b\Big\}dy\,. \qquad \square
\end{aligned}
$$

2.6 Transmission boundary conditions

We adopt the notation of Section 1.6.1. Let $M = (M_+, M_-)$ be a pair of compact manifolds with common smooth boundary

$$\Sigma = \partial M_+ = \partial M_- \,.$$

A structure S over M is a pair $S = (S_+, S_-)$ where S_+ and S_- are corresponding structures over M_+ and over M_-, respectively. Let $g = (g_+, g_-)$ be a Riemannian metric on M and let $V = (V_+, V_-)$ be a smooth vector bundle over M. Assume the compatibility conditions

$$g_+|_\Sigma = g_-|_\Sigma \qquad \text{and} \qquad V_-|_\Sigma = V_+|_\Sigma \,. \tag{2.6.a}$$

Let $D = (D_-, D_+)$ be an operator of Laplace type on V. Let $\nabla = (\nabla_+, \nabla_-)$ and $E = (E_+, E_-)$ be the associated connection and endomorphism given by Lemma 1.2.1. Let $\nu_\pm$ be the inward unit normals of $\Sigma \subset M_\pm$; $\nu_+ + \nu_- = 0$. Let

$$\omega_a := \nabla_a^+ - \nabla_a^- \qquad \text{on} \qquad V|_\Sigma;$$

this chiral tensor changes sign if we interchange the roles of M_+ and M_-. Let $\phi := (\phi_+, \phi_-)$ and $\rho := (\rho_+, \rho_-)$ be smooth sections to the vector bundles V and V^* over M, respectively. Suppose that there is given an auxiliary endomorphism U of $V_\Sigma := V_\pm|_\Sigma$ serving as an impedance matching term. We adopt the notation of Equation (1.6.a) and let

$$\mathcal{B}_U\phi := \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \ \oplus \ \{\nabla_\nu\phi_+|_\Sigma + \nabla_\nu\phi_-|_\Sigma - U\phi_+|_\Sigma\}$$

define *transmission boundary conditions*; ϕ satisfies these boundary conditions if and only if ϕ extends continuously across the interface Σ and if the normal derivatives match, modulo the impedance matching term U. This boundary condition arises in the study of heat conduction problems between closely coupled membranes as was discussed in Section 1.6.4.

The following theorem is due to Gilkey and Kirsten [194].

Theorem 2.6.1 *Adopt the notation established above.*

1. $\beta_0(\phi, \rho, D, \mathcal{B}_U) = \int_{M_+} \langle\phi_+, \rho_+\rangle dx_+ + \int_{M_-} \langle\phi_-, \rho_-\rangle dx_-$.

2. $\beta_1(\phi, \rho, D, \mathcal{B}_U) = -\frac{1}{\sqrt{\pi}} \int_\Sigma \langle\phi_+ - \phi_-, \rho_+ - \rho_-\rangle dy$.

3. $\beta_2(\phi, \rho, D, \mathcal{B}_U) = -\int_{M_+} \langle D_+\phi_+, \rho_+\rangle dx_+ - \int_{M_-} \langle D_-\phi_-, \rho_-\rangle dx_-$

$$\begin{aligned}
&+ \int_\Sigma \big\{ \tfrac{1}{8}(L_{aa}^+ + L_{aa}^-)(\langle\phi_+, \rho_+\rangle + \langle\phi_-, \rho_-\rangle) \\
&- \tfrac{1}{8}(L_{aa}^+ + L_{aa}^-)(\langle\phi_+, \rho_-\rangle + \langle\phi_-, \rho_+\rangle) \\
&+ \tfrac{1}{2}(\langle\phi_{+;\nu}, \rho_+\rangle + \langle\phi_{-;\nu}\rho_-\rangle + \langle\phi_{+;\nu}, \rho_-\rangle + \langle\phi_{-;\nu}, \rho_+\rangle) \\
&- \tfrac{1}{2}(\langle\phi_+, \rho_{+;\nu}\rangle + \langle\phi_-, \rho_{-;\nu}\rangle) + \tfrac{1}{2}(\langle\phi_+, \rho_{-;\nu}\rangle + \langle\phi_-, \rho_{+;\nu}\rangle) \\
&- \tfrac{1}{4}(\langle U\phi_+, \rho_+\rangle + \langle U\phi_-, \rho_-\rangle + \langle U\phi_+, \rho_-\rangle + \langle U\phi_-, \rho_+\rangle)\big\}.
\end{aligned}$$

4. $\beta_3(\phi,\rho,D,\mathcal{B}_U)$

$$
\begin{aligned}
&= \tfrac{1}{6\sqrt{\pi}}\int_\Sigma\{4(\langle D_+\phi_+,\rho_+\rangle+\langle\phi_+,\tilde D_+\rho_+\rangle+\langle D_-\phi_-,\rho_-\rangle+\langle\phi_-,\tilde D_-\rho_-\rangle)\\
&-4(\langle D_+\phi_+,\rho_-\rangle+\langle\phi_+,\tilde D_-\rho_-\rangle+\langle D_-\phi_-,\rho_+\rangle+\langle\phi_-,\tilde D_+\rho_+\rangle)\\
&-(\langle\omega_a\phi_{+;a},\rho_+\rangle-\langle\omega_a\phi_{-;a},\rho_-\rangle-\langle\omega_a\phi_+,\rho_{+;a}\rangle+\langle\omega_a\phi_-,\rho_{-;a}\rangle)\\
&-(\langle\omega_a\phi_{+;a},\rho_-\rangle-\langle\omega_a\phi_{-;a},\rho_+\rangle+\langle\omega_a\phi_+,\rho_{-;a}\rangle-\langle\omega_a\phi_-,\rho_{+;a}\rangle)\\
&+4(\langle\phi_{+;\nu},\rho_{+;\nu}\rangle+\langle\phi_{-;\nu},\rho_{-;\nu}\rangle+\langle\phi_{+;\nu},\rho_{-;\nu}\rangle+\langle\phi_{-;\nu},\rho_{+;\nu}\rangle)\\
&-2(\langle\phi_{+;a},\rho_{+;a}\rangle+\langle\phi_{-;a},\rho_{-;a}\rangle)\ +2(\langle\phi_{+;a},\rho_{-;a}\rangle+\langle\phi_{-;a},\rho_{+;a}\rangle)\\
&-2(\langle U\phi_{+;\nu},\rho_+\rangle+\langle U\phi_+,\rho_{+;\nu}\rangle+\langle U\phi_{-;\nu},\rho_-\rangle+\langle U\phi_-,\rho_{-;\nu}\rangle)\\
&-2(\langle U\phi_{-;\nu},\rho_+\rangle+\langle U\phi_-,\rho_{+;\nu}\rangle+\langle U\phi_{+;\nu},\rho_-\rangle+\langle U\phi_+,\rho_{-;\nu}\rangle)\\
&+(L_{aa}^--L_{aa}^+)(\nu_+\langle\phi_+,\rho_+\rangle-\nu_-\langle\phi_-,\rho_-\rangle)\\
&+L_{aa}^+(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)+L_{aa}^-(\langle\phi_{-;\nu},\rho_+\rangle+\langle\phi_+,\rho_{-;\nu}\rangle)\\
&-(L_{aa}^-(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)+L_{aa}^+(\langle\phi_{-;\nu},\rho_+\rangle+\langle\phi_+,\rho_{-;\nu}\rangle))\\
&+\langle\omega_a\omega_a\phi_+,\rho_+\rangle+\langle\omega_a\omega_a\phi_-,\rho_-\rangle\ -\tfrac12L_{aa}^+L_{bb}^-(\langle\phi_+,\rho_+\rangle+\langle\phi_-,\rho_-\rangle)\\
&+\tfrac12L_{aa}^+L_{bb}^-(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\ +\tfrac12(L_{ab}^+L_{ab}^+\langle\phi_+,\rho_+\rangle+L_{ab}^-L_{ab}^-\langle\phi_-,\rho_-\rangle)\\
&+\tfrac12(L_{ab}^-L_{ab}^-\langle\phi_+,\rho_+\rangle+L_{ab}^+L_{ab}^+\langle\phi_-,\rho_-\rangle)\\
&-\tfrac12(L_{ab}^+L_{ab}^++L_{ab}^-L_{ab}^-)(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\\
&+L_{aa}^+\langle U\phi_+,\rho_+\rangle+L_{aa}^-\langle U\phi_-,\rho_-\rangle-L_{aa}^-\langle U\phi_+,\rho_+\rangle-L_{aa}^+\langle U\phi_-,\rho_-\rangle\\
&+\langle U^2\phi_+,\rho_+\rangle+\langle U^2\phi_-,\rho_-\rangle+\langle U^2\phi_+,\rho_-\rangle+\langle U^2\phi_-,\rho_+\rangle\\
&+\langle E_+\phi_+,\rho_+\rangle+\langle E_-\phi_-,\rho_-\rangle+\langle E_-\phi_+,\rho_+\rangle+\langle E_+\phi_-,\rho_-\rangle\\
&-\langle(E_++E_-)\phi_+,\rho_-\rangle-\langle(E_++E_-)\phi_-,\rho_+\rangle\\
&+\tfrac12(R_{amma}^++R_{amma}^-)(\langle\phi_+,\rho_+\rangle+\langle\phi_-,\rho_-\rangle)\\
&-\tfrac12(R_{amma}^++R_{amma}^-)(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\}dy.
\end{aligned}
$$

Assertion (1) of Theorem 2.6.1 follows from Lemma 2.1.1. We shall follow the discussion in [194] in proving Assertions (2) and (3) in this book. As the proof of Assertion (4) is similar, and somewhat lengthy, we shall omit details and instead refer to [194] for further details.

2.6.1 Proof of Theorem 2.6.1 (2)

The manifolds M_+ and M_- play symmetric roles; the invariants β_n must therefore reflect this symmetry. We use Lemma 2.1.1 and Theorem 2.1.12 to see that there exist universal constants so

$$
\begin{aligned}
&\beta_1(\phi,\rho,D,\mathcal{B}_U) \qquad (2.6.b)\\
&\quad=\int_\Sigma\left\{c_1(\langle\phi_+,\rho_+\rangle+\langle\phi_-,\rho_-\rangle)+c_2(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\right\}dy\,.
\end{aligned}
$$

We adopt the notation of Lemma 2.2.1 and set

$$M_+=M_-=M_0,\ D_+=D_-=D_0,\qquad\text{and}\qquad U=-2S \qquad (2.6.c)$$

Let $\mathcal{B}_D$ define Dirichlet boundary conditions and let $\mathcal{B}_{R(S)}$ define Robin boundary conditions. We use Lemma 2.2.1, Theorem 2.3.3, and Theorem 2.4.1 to see that

$$\begin{aligned}\beta_1(\phi,\rho,D,\mathcal{B}_U) &= 2\beta_1(\phi_{\text{odd}}\,,\rho_{\text{odd}}\,,D,\mathcal{B}_D)+2\beta_1(\phi_{\text{even}}\,,\rho_{\text{even}}\,,D,\mathcal{B}_{R(S)})\\ &= -\tfrac{2}{\sqrt{\pi}}\int_\Sigma 2\langle\phi_{\text{odd}}\,,\rho_{\text{odd}}\,\rangle dy\\ &= -\tfrac{1}{\sqrt{\pi}}\int_{\partial M_0}\langle\phi_+-\phi_-,\rho_+-\rho_-\rangle dy\,. \end{aligned}\tag{2.6.d}$$

Equations (2.6.b) and (2.6.d) then imply

$$c_1=-\tfrac{1}{\sqrt{\pi}}\quad\text{and}\quad c_2=\tfrac{1}{\sqrt{\pi}}\,.$$

This establishes Assertion (2) of Theorem 2.6.1. □

2.6.2 Proof of Theorem 2.6.1 (3)

A similar argument shows there are universal constants so that

$$\beta_2(\phi,\rho,D,\mathcal{B}_U)=-\int_{M_+}\langle D_+\phi_+,\rho_+\rangle dx_+-\int_{M_-}\langle D_-\phi_-,\rho_-\rangle dx_-\tag{2.6.e}$$

$$\begin{aligned}&+\int_\Sigma\Big\{c_3(L_{aa}^+\langle\phi_+,\rho_+\rangle+L_{aa}^-\langle\phi_-,\rho_-\rangle)+c_4(L_{aa}^-\langle\phi_+,\rho_+\rangle+L_{aa}^+\langle\phi_-,\rho_-\rangle)\\ &\quad+c_5(L_{aa}^+\langle\phi_+,\rho_-\rangle+L_{aa}^-\langle\phi_-,\rho_+\rangle)+c_6(L_{aa}^-\langle\phi_+,\rho_-\rangle+L_{aa}^+\langle\phi_-,\rho_+\rangle)\\ &\quad+c_7(\langle\phi_{+;\nu},\rho_+\rangle+\langle\phi_{-;\nu},\rho_-\rangle)+c_8(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_{-;\nu},\rho_+\rangle)\\ &\quad+c_9(\langle\phi_+,\rho_{+;\nu}\rangle+\langle\phi_-,\rho_{-;\nu}\rangle)+c_{10}(\langle\phi_+,\rho_{-;\nu}\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)\\ &\quad+c_{11}(\langle U\phi_+,\rho_+\rangle+\langle U\phi_-,\rho_-\rangle)+c_{12}(\langle U\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\Big\}dy\,.\end{aligned}$$

Assertion (3) of Theorem 2.6.1 will then follow from the following:

Lemma 2.6.2

$$\begin{array}{lllll} c_3=\frac18, & c_4=\frac18, & c_5=-\frac18, & c_6=-\frac18, & c_7=\frac12,\\ c_8=\frac12, & c_9=-\frac12, & c_{10}=\frac12, & c_{11}=-\frac14, & c_{12}=-\frac14\,.\end{array}$$

Proof: We use Lemma 2.1.3 to see

$$0=\beta_2(\phi,\rho,D,\mathcal{B}_U)-\beta_2(\rho,\phi,\tilde{D},\widetilde{\mathcal{B}_U})\tag{2.6.f}$$

$$\begin{aligned}&=\int_M\Big\{-\langle D\phi,\rho\rangle+\langle\phi,\tilde{D}\rho\rangle\Big\}dx\\ &+\int_{\partial M}\Big\{(c_5-c_6)(L_{aa}^+-L_{aa}^-)(\langle\phi_+,\rho_-\rangle-\langle\phi_-,\rho_+\rangle)\\ &+(c_7-c_9)(\langle\phi_{+;\nu},\rho_+\rangle+\langle\phi_{-;\nu},\rho_-\rangle-\langle\phi_+,\rho_{+;\nu}\rangle-\langle\phi_-,\rho_{-;\nu}\rangle)\\ &+(c_8-c_{10})(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_{-;\nu},\rho_+\rangle-\langle\phi_+,\rho_{-;\nu}\rangle-\langle\phi_-,\rho_{+;\nu}\rangle)\Big\}dy\,.\end{aligned}$$

By Theorem 1.4.17,

$$\begin{aligned} 0 &= \int_M \Big\{ -\langle D\phi, \rho\rangle + \langle \phi, \tilde{D}\rho\rangle \Big\} dx \\ &- \int_\Sigma \Big\{ \langle \phi_+, \rho_{+;\nu}\rangle + \langle \phi_-, \rho_{-;\nu}\rangle - \langle \phi_{+;\nu}, \rho_+\rangle - \langle \phi_{-;\nu}, \rho_-\rangle \Big\} dy\,. \end{aligned} \tag{2.6.g}$$

We use Equations (2.6.f) and Equation (2.6.g) to derive the relations

$$c_5 = c_6,\ c_7 - c_9 = 1,\ c_8 = c_{10}\,. \tag{2.6.h}$$

Adopt the notation of Equation (2.6.c). By Lemma 2.2.1,

$$\beta_2(\phi, \rho, D, \mathcal{B}_U) = 2\beta_2(\phi_{\text{odd}}, \rho_{\text{odd}}, D_0, \mathcal{B}_D) + 2\beta_2(\phi_{\text{even}}, \rho_{\text{even}}, D_0, \mathcal{B}_{R(S)}).$$

Lemma 2.2.1, Theorem 2.3.3, and Theorem 2.4.1 then imply:

$$\begin{aligned} &\beta_2(\phi, \rho, D, \mathcal{B}_U) = 2\beta_2(\phi_{\text{odd}}, \rho_{\text{odd}}, D_0, \mathcal{B}_D) \\ &\quad +2\beta_2(\phi_{\text{even}}, \rho_{\text{even}}, D_0, \mathcal{B}_{R(S)}) \\ &= \int_M \Big\{ -2\langle D\phi_{\text{odd}}, \rho_{\text{odd}}\rangle - 2\langle D\phi_{\text{even}}, \rho_{\text{even}}\rangle \Big\} dx \qquad (2.6.\text{i}) \\ &+ \int_\Sigma \Big\{ L_{aa}\langle \phi_{\text{odd}}, \rho_{\text{odd}}\rangle - 2\langle \phi_{\text{odd}}, \rho_{\text{odd};\nu}\rangle \\ &\quad +2\langle \phi_{\text{even};\nu}, \rho_{\text{even}}\rangle + \langle 2S\phi_{\text{even}}, \rho_{\text{even}}\rangle \Big\} dy\,. \end{aligned}$$

The interior integrals in Equations (2.6.e) and (2.6.i) agree. Comparing the boundary integrands yields the relations:

$$\begin{array}{ll} 2c_3 + 2c_4 + 2c_5 + 2c_6 = 0, & 2c_7 + 2c_8 = 2, \\ 2c_9 + 2c_{10} = 0, & -4c_{11} - 4c_{12} = 2 \\ 2c_3 + 2c_4 - 2c_5 - 2c_6 = 1, & 2c_7 - 2c_8 = 0, \\ 2c_9 - 2c_{10} = -2, & 2c_{11} - 2c_{12} = 0. \end{array} \tag{2.6.j}$$

We return to the general setting. Let (M_+, M_-) and (g_+, g_-) be arbitrary subject to the compatibility condition $g_+|_\Sigma = g_-|_\Sigma$. Let $D_\pm$ be the scalar Laplacians on $M_\pm$, respectively. Take $\phi = 1$ and $U = 0$. By Lemma 2.2.2,

$$\beta_2(\phi, \rho, D, \mathcal{B}_U) = 0$$

in this setting. Take $\rho_- = 0$. The terms

$$\{\rho_+,\ \rho_+ L_{aa}^+,\ \rho_+ L_{aa}^-,\ \rho_{+;\nu}\}$$

can then be specified arbitrarily. This yields the relations

$$c_3 + c_6 = 0,\ c_4 + c_5 = 0,\ c_9 + c_{10} = 0\,. \tag{2.6.k}$$

Lemma 2.6.2, and consequently Assertion (3) of Theorem 2.6.1, now follows from the relations in Displays (2.6.h), (2.6.j) and (2.6.k). □

2.7 Transfer boundary conditions

As in Section 2.6, let

$$M = (M_+, M_-), \quad V = (V_+, V_-), \quad \text{and} \quad D = (D_+, D_-)$$

where $D_\pm$ are operators of Laplace type on bundles $V_\pm$ over compact Riemannian manifolds $M_\pm$ with common boundary $\partial M_+ = \partial M_- = \Sigma$. In place of the compatibility condition given in Equation (2.6.a), we instead assume only

$$g_+|_\Sigma = g_-|_\Sigma \quad . \tag{2.7.a}$$

We emphasize that we do **not** assume an identification of V_+ with V_- over Σ. In particular, we can consider the case $\dim(V_1) \neq \dim(V_2)$. Let $\nu_\pm$ be the inward unit normals of Σ in $M_\pm$; $\nu_+ + \nu_- = 0$.

In the previous section, we studied transmission boundary conditions; physically, as was discussed in Section 1.6.4, this corresponds to having the two components pressed tightly together. By contrast, in this section, we shall study *heat transfer boundary conditions* $\mathcal{B}_\mathcal{S}$; this corresponds to a loose coupling between the two components. We adopt the notation of Section 1.6.3 and define:

$$\begin{aligned}\mathcal{B}_\mathcal{S}\phi &:= \left\{\begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--}\end{pmatrix}\begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix}\right\}\Bigg|_\Sigma, \\ \mathcal{B}_{\tilde{\mathcal{S}}}\rho &:= \left\{\begin{pmatrix} \tilde\nabla^+_{\nu_+} + \tilde S_{++} & \tilde S_{-+} \\ \tilde S_{+-} & \tilde\nabla^-_{\nu_-} + \tilde S_{--}\end{pmatrix}\begin{pmatrix}\rho_+ \\ \rho_-\end{pmatrix}\right\}\Bigg|_\Sigma .\end{aligned} \tag{2.7.b}$$

where

$$\begin{aligned} S_{++} &: V_+|_\Sigma \to V_+|_\Sigma, \qquad & S_{+-} &: V_-|_\Sigma \to V_+|_\Sigma, \\ S_{-+} &: V_+|_\Sigma \to V_-|_\Sigma, \qquad & S_{--} &: V_-|_\Sigma \to V_-|_\Sigma .\end{aligned}$$

The following theorem is due to Gilkey and Kirsten [194].

Theorem 2.7.1 *Adopt the notation established above.*

1. $\beta_0(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = \int_{M_+}\langle\phi_+,\rho_+\rangle dx_+ + \int_{M_-}\langle\phi_-,\rho_-\rangle dx_-$.
2. $\beta_1(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = 0$.
3. $\beta_2(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = -\int_{M_+}\langle D_+\phi_+,\rho_+\rangle dx_+ - \int_{M_-}\langle D_-\phi_-,\rho_-\rangle dx_-$
   $+\int_\Sigma\langle\mathcal{B}_\mathcal{S}\phi,\rho\rangle dy$.
4. $\beta_3(\phi,\rho,D,\mathcal{B}_\mathcal{S}) = \frac{4}{3\sqrt{\pi}}\int_\Sigma\langle\mathcal{B}_\mathcal{S}\phi,\mathcal{B}_{\tilde{\mathcal{S}}}\rho\rangle)dy$.

2.7.1 Proof of Theorem 2.7.1 (1-3)

We introduce the associated Robin boundary operators by defining

$$\mathcal{B}_{R(S_{++})} := (\nabla_{\nu_+} + S_{++}) \quad \text{and} \quad \mathcal{B}_{R(S_{--})} := (\nabla_{\nu_-} + S_{--}) .$$

By Lemma 2.2.5, if $S_{-+} = S_{+-} = 0$, then

$$\beta_n(\phi, \rho, D, \mathcal{B}) = \beta_n(\phi_+, \rho_+, D_+, \mathcal{B}_{R(S_{++})}) + \beta_n(\phi_-, \rho_-, D_-, \mathcal{B}_{R(S_{--})}) .$$

Thus the additional interaction terms must involve either S_{+-} or S_{-+}, or their tangential covariant derivatives. Since we have not assumed any relationship between $V_+|_\Sigma$ and $V_-|_\Sigma$, there can be no interaction between $\phi_\pm$ and $\rho_\mp$ which is not mediated by such a term. Assertions (1) and (2) of Theorem 2.7.1 now follow from the corresponding assertions of Theorem 2.4.1; such terms can not appear in the boundary integrals for β_0 and β_1 by Theorem 2.1.12 since S has weight 1.

Furthermore, only the interactions $\langle S_{-+}\phi_+, \rho_-\rangle$ and $\langle S_{+-}\phi_-, \rho_+\rangle$ are possible in the boundary integrand for β_2. As the roles of M_+ and M_- are symmetric, we conclude that there is a universal constant c_0 so that

$$\begin{aligned} \beta_2(\phi, \rho, D, \mathcal{B}_\mathcal{S}) = &- \int_{M_+} \langle D_+\phi_+, \rho_+\rangle dx_+ - \int_{M_-} \langle D_-\phi_-, \rho_-\rangle dx_- \\ &+ \int_\Sigma \Big\{ \langle \mathcal{B}_\mathcal{S}\phi, \rho\rangle + c_0(\langle S_{-+}\phi_+, \rho_-\rangle + \langle S_{+-}\phi_-, \rho_+\rangle \Big\} dy . \end{aligned}$$

Let $D_\pm$ be the scalar Laplacians on the manifolds $M_\pm$. Let

$$S_{++} = S_{--} = 1, \quad S_{+-} = S_{-+} = -1, \quad \text{and} \quad \phi = 1 .$$

Then, by Lemma 2.2.4,

$$\beta_2(\phi, \rho, D, \mathcal{B}_\mathcal{S}) = 0 .$$

We take $\rho_+ = 0$ and $\rho_- = 1$. Since $\langle S_{-+}\phi_+, \rho_-\rangle \neq 0$ while $\langle \mathcal{B}_\mathcal{S}\phi, \rho\rangle = 0$,

$$c_0 = 0 . \qquad \square$$

2.7.2 Proof of Theorem 2.7.1 (4)

A similar argument shows that there are universal constants so

$$\begin{aligned} \beta_3(\phi, \rho, D, \mathcal{B}_\mathcal{S}) = \tfrac{4}{3\sqrt{\pi}} \int_\Sigma \Big\{ &\langle \mathcal{B}_\mathcal{S}\phi, \mathcal{B}_{\tilde{\mathcal{S}}}\rho\rangle \\ &+c_1(\langle S_{+-}S_{-+}\phi_+, \rho_+\rangle + \langle S_{-+}S_{+-}\phi_-, \rho_-\rangle) \\ &+c_2(\langle S_{--}S_{-+}\phi_+, \rho_-\rangle + \langle S_{++}S_{+-}\phi_-, \rho_+\rangle) \\ &+c_3(\langle S_{-+}S_{++}\phi_+, \rho_-\rangle + \langle S_{+-}S_{--}\phi_-, \rho_+\rangle) \\ &+c_4(\langle S_{-+}\phi_{+;\nu_+}, \rho_-\rangle + \langle S_{+-}\phi_{-;\nu_-}, \rho_+\rangle) \\ &+c_5(\langle S_{-+}\phi_+, \rho_{-;\nu_-}\rangle + \langle S_{+-}\phi_-, \rho_{+;\nu_+}\rangle) \\ &+c_6(L^+_{aa}\langle S_{-+}\phi_+, \rho_-\rangle + L^-_{aa}\langle S_{+-}\phi_-, \rho_+\rangle) \\ &+c_7(L^-_{aa}\langle S_{-+}\phi_+, \rho_-\rangle + L^+_{aa}\langle S_{+-}\phi_-, \rho_+\rangle) \Big\} dy . \end{aligned}$$

We complete the proof of Assertion (4) of Theorem 2.7.1 by showing all these constants vanish; the requisite interactions involving S_{+-} and S_{-+} having already encoded into the term $\langle \mathcal{B}_\mathcal{S}\phi, \mathcal{B}_{\tilde{\mathcal{S}}}\rho\rangle$.

Let $D_\pm$ be the scalar Laplacians on manifolds $M_\pm$. Let

$$\phi = 1, \qquad S_{++} = -S_{+-} = a, \qquad \text{and} \qquad S_{--} = -S_{-+} = b\,.$$

Then $\beta_3(\phi, \rho, D, \mathcal{B}_\mathcal{S}) = 0$ by Lemma 2.2.4. We set $\rho_- = 0$ to see

$$c_1 - c_3 = 0 \quad \text{and} \quad c_2 = c_5 = c_6 = c_7 = 0. \tag{2.7.c}$$

The symmetry $\beta_3(\phi, \rho, D, \mathcal{B}_\mathcal{S}) = \beta_3(\rho, \phi, \tilde{D}, \mathcal{B}_{\tilde{\mathcal{S}}})$ of Lemma 2.1.3 yields

$$c_2 = c_3 \qquad \text{and} \qquad c_4 = c_5\,. \tag{2.7.d}$$

We combine Displays (2.7.c) and (2.7.d) to see all the universal coefficients vanish. This completes the proof. □

2.8 Oblique boundary conditions

Let D be an operator of Laplace type on a bundle V over M. Let $\mathcal{B}_T$ be a tangential first order partial differential operator on $V|_{\partial M}$. The associated *oblique boundary conditions* for D are defined by the operator

$$\mathcal{B}\phi := (\phi_{;m} + \mathcal{B}_T\phi)|_{\partial M}.$$

By Lemma 1.6.8, the dual boundary condition is defined by

$$\tilde{\mathcal{B}}\rho := (\rho_{;m} + \tilde{\mathcal{B}}_T\rho)|_{\partial M}$$

where $\tilde{\mathcal{B}}_T$ is the dual tangential boundary operator. By Lemma 1.6.8, $(D, \mathcal{B})$ is elliptic with respect to a suitable cone $\mathcal{C}_\delta$ provided that the leading symbol of $\mathcal{B}_T$ is small and hence is *admissible*. Consequently $e^{-tD_\mathcal{B}}$ and hence the heat content asymptotics will be well defined. Note that we recover Robin boundary conditions by taking $\mathcal{B}_T$ to be a 0^{th} order operator.

The following theorem, which generalizes Theorem 2.4.1, is due to Gilkey, Kirsten, and Park [196]; we shall follow the discussion there for the proof.

Theorem 2.8.1 *Let D be an operator of Laplace type on a vector bundle V defined over a compact Riemannian manifold with smooth boundary ∂M. Let $\mathcal{B}$ define oblique boundary conditions. Assume $(D, \mathcal{B})$ is admissible. Then:*

1. $\beta_0(\phi, \rho, D, \mathcal{B}) = \int_M \langle \phi, \rho \rangle dx$.

2. $\beta_1(\phi, \rho, D, \mathcal{B}) = 0$.

3. $\beta_2(\phi, \rho, D, \mathcal{B}) = -\int_M \langle D\phi, \rho \rangle dx + \int_{\partial M} \langle \mathcal{B}\phi, \rho \rangle dy$.

4. $\beta_3(\phi, \rho, D, \mathcal{B}) = \frac{4}{3\sqrt{\pi}} \int_{\partial M} \langle \mathcal{B}\phi, \tilde{\mathcal{B}}\rho \rangle dy$.

5. $\beta_4(\phi, \rho, D, \mathcal{B}) = \frac{1}{2} \int_M \langle D\phi, \tilde{D}\rho \rangle dx + \int_{\partial M} \{-\frac{1}{2}\langle \mathcal{B}\phi, \tilde{D}\rho \rangle - \frac{1}{2}\langle D\phi, \tilde{\mathcal{B}}\rho \rangle$
$+\langle (\frac{1}{2}\mathcal{B}_T + \frac{1}{4}L_{aa})\mathcal{B}\phi, \tilde{\mathcal{B}}\rho \rangle\} dy$.

This Theorem extends Theorem 2.4.1 by replacing the auxiliary term S in the Robin boundary operator by a more general first order tangential partial differential operator $\mathcal{B}_T$.

Assertion (1) follows from Lemma 2.1.1. The primary new difficulty, which arises in establishing the remaining assertions of the Theorem, is that the leading symbol of $\mathcal{B}_T$ has weight 0 and thus the dependence upon these variables in various coefficients is not controlled by Theorem 2.1.12. We use Lemma 2.1.1 to determine the interior integrands; as previously, keeping in mind Lemma 2.1.3, we replace the interior integrand $-\langle D^2\phi,\rho\rangle$ of β_4 by the more symmetric integrand $-\langle D\phi,\tilde{D}\rho\rangle$. Motivated by the formulae of Theorem 2.4.1, we express

$$\begin{aligned}
\beta_0(\phi,\rho,D,\mathcal{B}) &= \int_{\partial M}\langle\phi,\rho\rangle dx,\\
\beta_1(\phi,\rho,D,\mathcal{B}) &= \int_{\partial M}\mathcal{E}_1(\phi,\rho,D,\mathcal{B})dy\\
\beta_2(\phi,\rho,D,\mathcal{B}) &= -\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\Big\{\langle\mathcal{B}\phi,\rho\rangle+\mathcal{E}_2(\phi,\rho,D,\mathcal{B})\Big\}dy\\
\beta_3(\phi,\rho,D,\mathcal{B}) &= \int_{\partial M}\Big\{\tfrac{4}{3\sqrt{\pi}}\langle\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle+\mathcal{E}_3(\phi,\rho,D,\mathcal{B})\Big\}dy\\
\beta_4(\phi,\rho,D,\mathcal{B}) &= \tfrac12\int_M\langle D\phi,\tilde{D}\rho\rangle dx+\int_{\partial M}\Big\{-\tfrac12\langle\mathcal{B}\phi,\tilde{D}\rho\rangle-\tfrac12\langle D\phi,\tilde{\mathcal{B}}\rho\rangle\\
&\qquad+\langle(\tfrac12\mathcal{B}_T+\tfrac14L_{aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle+\mathcal{E}_4(\phi,\rho,D,\mathcal{B})\Big\}dy\,.
\end{aligned}$$

By Theorem 2.4.1, $\mathcal{E}_i(\phi,\rho,D,\mathcal{B})=0$ if $\mathcal{B}_T$ is a 0^{th} order operator. Set $\mathcal{E}_n=0$ for $n\le 0$.

Lemma 2.8.2 *Adopt the notation established above. Let $n\le 4$. Then:*

1. $\int_{\partial M}\mathcal{E}_n(\phi,\rho,D,\mathcal{B})dy=\int_{\partial M}\mathcal{E}_n(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})dy$.
2. *If* $\mathcal{B}\phi=0$, *then* $\frac{n}{2}\int_{\partial M}\mathcal{E}_n(\phi,\rho,D,\mathcal{B})dy=-\int_{\partial M}\mathcal{E}_{n-2}(D\phi,\rho,D,\mathcal{B})dy$.
3. *If* $\tilde{\mathcal{B}}\rho=0$, *then* $\frac{n}{2}\int_{\partial M}\mathcal{E}_n(\phi,\rho,D,\mathcal{B})dy=-\int_{\partial M}\mathcal{E}_{n-2}(\phi,\tilde{D}\rho,D,\mathcal{B})dy$.

Proof: By Lemma 2.1.3, $\beta_n(\phi,\rho,D,\mathcal{B})=\beta_n(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})$. For $n\ne 2$, the integrands which do not involve $\mathcal{E}_n$ are symmetric in the roles of V and V^* and thus Assertion (1) holds. If $n=2$, the individual integrands no longer play symmetric roles for V and V^*. However, by Equation (1.6.o),

$$\begin{aligned}
-\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\langle\mathcal{B}\phi,\rho\rangle dy &= -\int_M\langle\phi,\tilde{D}\rho\rangle dx+\int_{\partial M}\langle\phi,\tilde{\mathcal{B}}\rho\rangle dy \quad\text{so}\\
\int_{\partial M}\mathcal{E}_2(\phi,\rho,D,\mathcal{B})dy &= \int_{\partial M}\mathcal{E}_2(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})dy\,.
\end{aligned}$$

This completes the proof of Lemma 2.8.2 (1).

Suppose that $\tilde{\mathcal{B}}\rho = 0$. By Lemma 2.1.4,

$$\tfrac{n}{2}\beta_n(\rho, \phi, \tilde{D}, \tilde{\mathcal{B}}) = -\beta_{n-2}(\tilde{D}\rho, \phi, \tilde{D}, \tilde{\mathcal{B}})\,.$$

Assertion (3) now follows by inspection. Assertion (2) follows from Assertions (1) and (3). $\square$

Suppose either that $\mathcal{B}\phi = 0$ or that $\tilde{\mathcal{B}}\rho = 0$. Because $\mathcal{E}_{-1} = 0$ and $\mathcal{E}_0 = 0$, we use Lemma 2.8.2 to see

$$\int_{\partial M} \mathcal{E}_n(\phi, \rho, D, \mathcal{B})dy = 0 \quad \text{for} \quad n \le 2 \;\text{ if }\; \mathcal{B}\phi = 0 \;\text{ or }\; \tilde{\mathcal{B}}\rho = 0\,. \tag{2.8.a}$$

We reparametrize the Cauchy data to use $\{\phi|_{\partial M}, \mathcal{B}\phi\}$ and $\{\rho|_{\partial M}, \tilde{\mathcal{B}}\rho\}$ as a basis for expressing the invariants of total weight at most 1 on the boundary. This leads to expressions of the form

$$\begin{aligned}
&\mathcal{E}_1(\phi, \rho, D, \mathcal{B}) = \langle \mathcal{T}_{00}^1\phi, \rho\rangle,\\
&\mathcal{E}_2(\phi, \rho, D, \mathcal{B}) = \langle \mathcal{T}_{00}^2\phi, \rho\rangle + \langle \mathcal{T}_{10}^2\mathcal{B}\phi, \rho\rangle + \langle \mathcal{T}_{01}^2\phi, \tilde{\mathcal{B}}\rho\rangle
\end{aligned}$$

where $\mathcal{T}_{uv}^j$ are suitably chosen tangential operators. Since we can specify $\phi|_{\partial M}$, $\rho|_{\partial M}$, $\mathcal{B}\phi$, and $\tilde{\mathcal{B}}\rho$ arbitrarily, Equation (2.8.a) when $\mathcal{B}\phi = 0$ implies

$$\int_{\partial M}\langle \mathcal{T}_{00}^1\phi, \rho\rangle dy = 0,\; \int_{\partial M}\langle \mathcal{T}_{00}^2\phi, \rho\rangle dy = 0,\; \int_{\partial M}\langle \mathcal{T}_{01}^2\phi, \tilde{\mathcal{B}}\rho\rangle dy = 0\,.$$

Since, subject to the constraint $\mathcal{B}\phi = 0$, $\phi|_{\partial M}$ can be specified arbitrarily, this vanishing for **any** (ϕ, ρ) implies $T_{00}^1 = 0$, $\mathcal{T}_{00}^2 = 0$, and $T_{01}^2 = 0$. Similarly we use Equation (2.8.a) with $\tilde{\mathcal{B}}\rho = 0$ and argue similarly to see $T_{10}^2 = 0$. Assertions (2) and (3) of Theorem 2.8.1 now follow.

Since the integrals of $\mathcal{E}_n$ vanish for $n = 1, 2$, we now have that

$$\int_{\partial M} \mathcal{E}_n(\phi, \rho, D, \mathcal{B})dy = 0 \quad \text{for} \quad n \le 4 \;\text{ if }\; \mathcal{B}\phi = 0 \;\text{ or }\; \tilde{\mathcal{B}}\rho = 0\,. \tag{2.8.b}$$

A similar reparametrization of the Cauchy data map permits us to express:

$$\begin{aligned}
\mathcal{E}_3(\phi, \rho, D, \mathcal{B}) \;&=\; \int_{\partial M}\Big\{\langle \mathcal{T}_{00}^3\phi, \rho\rangle + \langle \mathcal{T}_{10}^3\mathcal{B}\phi, \rho\rangle + \langle \mathcal{T}_{20}^3\phi_{;mm}, \rho\rangle\\
&\quad + \;\langle \mathcal{T}_{11}^3\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle + \langle \mathcal{T}_{01}^3\phi, \tilde{\mathcal{B}}\rho\rangle + \langle \mathcal{T}_{02}^3\phi, \rho_{;mm}\rangle\Big\}dy\,.
\end{aligned}$$

We use Equation (2.8.b) and argue as above to see there exist tangential operators $\mathcal{T}_{11}^3$ and $\mathcal{T}_{11}^4$ of weights 1 and 2, respectively, so that

$$\mathcal{E}_3(\phi, \rho, D, \mathcal{B}) = \langle T_{11}^3\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle \quad\text{and}\quad \mathcal{E}_4(\phi, \rho, D, \mathcal{B}) = \langle \mathcal{T}_{11}^4\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle\,.$$

We decompose $\mathcal{B}_T\phi = \Gamma_a\nabla_{e_a} + S$ and express

$$\begin{aligned}
\mathcal{E}_3(\phi, \rho, D, \mathcal{B}) \;&=\; \int_{\partial M}\langle c_0(\Gamma)\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle dy\\
\mathcal{E}_4(\phi, \rho, D, \mathcal{B}) \;&=\; \int_{\partial M}\Big\{\langle (c_1(\Gamma, S) + c_2(\Gamma, L) + c_3(\Gamma, \nabla\Gamma)
\end{aligned}$$

$$+ \quad c_4^a(\Gamma)\nabla_{e_a})\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle \Big\} dy$$

where $c_1(\Gamma, S)$, $c_2(\Gamma, L)$, and $c_3(\Gamma, \nabla\Gamma)$ are linear in S, L, and $\nabla\Gamma$, respectively.

We adopt the notation of Lemma 2.1.11. Let

$$ds_M^2 := g_{uv}(r)d\theta^u \circ d\theta^v + dr^2 \quad \text{on} \quad M := \mathbb{T}^{m-1} \times [0,1]\,.$$

Let Δ be the associated scalar Laplacian. Let $V := M \times \mathbb{C}^\ell$. Let

$$D := \Delta \otimes \mathrm{Id} - A^a(\theta, r)\partial_a^\theta \quad \text{and} \quad \mathcal{B} := \partial_r \otimes \mathrm{Id} + B^a(\theta, r)\partial_a^\theta + S$$

where $S \in M_\ell(\mathbb{C})$ is constant and where $A^a, \Gamma^a \in C^\infty(\mathrm{End}\,(V))$ are arbitrary. Assume $(D, \mathcal{B})$ is admissible. Let $\phi = \phi(r)$ and $\rho = \rho(\theta, r)$. By Lemma 2.1.11, $\beta_n(\phi, \rho, D, \mathcal{B})$ is independent of $\{A^a, \Gamma^a\}$.

Let $\mathcal{B}_0$ be defined by setting $\Gamma = 0$. We can integrate by parts to see

$$\begin{aligned}
&\int_{\partial M}\langle \mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle dy = \int_{\partial M}\langle \mathcal{B}_0\phi, \tilde{\mathcal{B}}_0\rho\rangle dy, \quad \text{and}\\
&\int_{\partial M}\Big\{ -\tfrac12\langle \mathcal{B}\phi, \tilde{D}\rho\rangle - \tfrac12\langle D\phi, \tilde{\mathcal{B}}\rho\rangle + \langle(\tfrac12\mathcal{B}_T + \tfrac12 L_{aa})\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle\Big\} dy\\
= \;&\int_{\partial M}\Big\{ -\tfrac12\langle \mathcal{B}_0\phi, \tilde{D}\rho\rangle - \tfrac12\langle D\phi, \tilde{\mathcal{B}}_0\rho\rangle + \langle(\tfrac12 S + \tfrac12 L_{aa})\mathcal{B}_0\phi, \tilde{\mathcal{B}}_0\rho\rangle\Big\} dy\,.
\end{aligned}$$

Consequently, we conclude

$$\int_{\partial M}\langle T_{11}^3\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle dy = \int_{\partial M}\langle T_{11}^4\mathcal{B}\phi, \tilde{\mathcal{B}}\rho\rangle dy = 0\,.$$

Since $\tilde{\mathcal{B}}\rho$ can be chosen arbitrarily, we have $T_{11}^3\mathcal{B}\phi = 0$ and $T_{11}^4\mathcal{B}\phi = 0$ pointwise. This establishes the desired vanishing $T_{11}^3 = 0$ and $T_{11}^4 = 0$; we choose the auxiliary terms A^α appropriately to ensure $\nabla_{e_a}\phi$ can be specified arbitrarily. This also completes the proof of Theorem 2.8.1. □

Remark 2.8.3 For the heat trace asymptotics, the breakdown of the classic Lopatinskij condition of ellipticity for large values of Γ is clearly reflected in the heat kernel coefficients; they become singular at these values [25, 146]. It is therefore somewhat curious that this breakdown is **not** reflected in the heat content asymptotics; the formulae of Theorem 2.8.1 are well defined for **any** first order tangential operator $\mathcal{B}_T$.

2.9 Variable geometries

We adopt the notation of Section 1.6.9. Let $\mathfrak{g} := \{g_t\}$ be a smooth 1 parameter family of Riemannian metrics on M, and let $\mathfrak{D} := \{D_t\}$ be a smooth 1 parameter family operators of Laplace type on a bundle V over M with either Dirichlet or Robin boundary conditions.

Let dx and dy be the measures defined by the initial metric g_0. Let ∇ be the connection on V defined by the initial operator D_0 using Lemma 1.2.1. Let $\{e_1, ..., e_m\}$ be a local frame field for TM which is orthonormal with respect to the initial metric g_0. We may then define tensors $\mathcal{G}_{r,ij}$, $\mathcal{F}_{r,i}$ and $\mathcal{E}_r$ by expanding D_t in a Taylor series

$$D_t\phi = D_0\phi + \sum_{r=1}^{\infty} \left\{ \mathcal{G}_{r,ij}\phi_{;ij} + \mathcal{F}_{r,i}\phi_{;i} + \mathcal{E}_r\phi \right\} t^r \,. \tag{2.9.a}$$

This setting appears most naturally when defining an adiabatic vacuum in quantum field theory in curved spacetime [62]. If the spacetime is slowly varying, then the time-dependent metric describing the cosmological evolution can be expanded in a Taylor series with respect to t. The index r in this situation is then related to the adiabatic order. The following result corrects a minor mistake, noted by Park [299], in the computations of [190] where the coefficients of $L_{ab}\langle\mathcal{G}_{1,ab}\phi, \rho\rangle$, $L_{aa}\langle\mathcal{G}_{1,mm}\phi, \rho\rangle$, and $\langle\mathcal{G}_{1,am}\phi_{:a}, \rho\rangle$ in the boundary integral for β_4 were given incorrectly.

Theorem 2.9.1 *Adopt the notation established above. Let $\mathfrak{D} = \{D_t\}$ be a smooth 1 parameter family of operators of Laplace type with respect to a smooth 1 parameter family of metrics g_t. Let $\mathcal{B}$ define Dirichlet boundary conditions.*

1. $\beta_n(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_n(\phi, \rho, D_0, \mathcal{B})$ *for* $n = 0, 1, 2$.

2. $\beta_3(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_3(\phi, \rho, D_0, \mathcal{B}) + \frac{1}{2\sqrt{\pi}} \int_{\partial M} \langle\mathcal{G}_{1,mm}\phi, \rho\rangle dy$.

3. $\beta_4(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_4(\phi, \rho, D_0, \mathcal{B}) - \frac{1}{2} \int_M \langle\mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi, \rho\rangle dx$
$+ \int_{\partial M} \{ \frac{7}{16}\langle\mathcal{G}_{1,mm;m}\phi, \rho\rangle - \frac{9}{16} L_{aa}\langle\mathcal{G}_{1,mm}\phi, \rho\rangle - \frac{5}{16}\langle\mathcal{F}_{1,m}\phi, \rho\rangle$
$+ \frac{5}{16} L_{ab}\langle\mathcal{G}_{1,ab}\phi, \rho\rangle - \frac{5}{8}\langle\mathcal{G}_{1,am}\phi_{:a}, \rho\rangle + \frac{1}{2}\langle\mathcal{G}_{1,mm}\phi, \rho_{;m}\rangle \} dy$.

Let $\nu_t = \xi_i(t)e_i$ be the inward unit vector field defined by the metrics g_t where $\xi_m(0) = 1$ and $\xi_a(0) = 0$. We can replace the time-dependent Robin boundary operators

$$B_t\phi = \{\nabla_{\nu_t} + S(t)\}\phi|_{\partial M}$$

by an equivalent family of boundary operators

$$\check{B}_t\phi = \{\phi_{;m} + \xi_m^{-1}\xi_a\phi_{;a} + \xi_m^{-1}S(t)\phi\}|_{\partial M} \,.$$

This motivates the study of boundary conditions $\mathfrak{B} = \{\mathcal{B}_t\}$ where

$$\mathcal{B}_t\phi = \left\{ \phi_{;m} + S\phi + \sum_{r=1}^{\infty} t^r (\Gamma_{a,r}\nabla_{e_a} + S_r)\phi \right\} \Bigg|_{\partial M} \,. \tag{2.9.b}$$

Such boundary conditions are elliptic with respect to a suitable cone $\mathcal{C}_\delta$ for small t by Lemma 1.6.8. The reason for including a dependence on time in the boundary condition comes, for example, by considering the dynamical Casimir effect. Slowly moving boundaries give rise to such boundary conditions. The following result extends previous work in [190].

Theorem 2.9.2 *Let $\mathfrak{D}$ be a smooth 1 parameter family of operators of Laplace type. Let $\mathfrak{B}$ be boundary conditions of the form given in Equation (2.9.b). Then*

1. $\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_n(\phi,\rho,D_0,\mathcal{B}_0)$ *for* $n \le 3$.
2. $\beta_4(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_4(\phi,\rho,D_0,\mathcal{B}_0) - \frac{1}{2}\int_M \langle \mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi,\rho\rangle dx$
   $+ \int_{\partial M}\{-\frac{1}{2}\langle \mathcal{G}_{1,mm}\mathcal{B}_0\phi,\rho\rangle + \frac{1}{2}\langle (S_1 + \Gamma_a\nabla_{e_a})\phi,\rho\rangle\}dy.$

We shall follow the discussion in [190]. We assign weight $2r$ to $\mathcal{G}_r$, weight $2r+1$ to $\mathcal{F}_r$, weight $2r+2$ to $\mathcal{E}_r$, weight $2r$ to $\Gamma_{a,r}$, and weight $2r+1$ to S_r. By Remark 2.1.14, the interior invariants β_n^M are homogeneous of weight n while the boundary invariants $\beta_n^{\partial M}$ are homogeneous of weight $n-1$. Thus, in particular, these additional tensors, and their covariant derivatives, do not enter into the formulae for β_0 or β_1. Express the remaining invariants relative to a Weyl basis. Lemma 2.1.7 and Lemma 2.1.8 then show the constants relative to a Weyl basis are universal and depend neither on the rank of the vector bundle nor on the dimension of the underlying manifold.

2.9.1 The proof of Theorem 2.9.1

Impose Dirichlet boundary conditions. We begin by noting that:

Lemma 2.9.3 *There exist universal constants so that*

1. $\beta_2(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_2(\phi,\rho,D_0,\mathcal{B}) + \int_M \langle a_1\mathcal{G}_{1,ii}\phi,\rho\rangle dx.$
2. $\beta_3(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_3(\phi,\rho,D_0,\mathcal{B}) + \int_{\partial M}\{a_2\langle \mathcal{G}_{1,aa}\phi,\rho\rangle + b_1\langle \mathcal{G}_{1,mm}\phi,\rho\rangle\}dy.$
3. $\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_4(\phi,\rho,D_0,\mathcal{B}) + \int_M\{a_3\langle \mathcal{G}_{1,ii;jj}\phi,\rho\rangle + a_4\langle \mathcal{G}_{1,ij;ij}\phi,\rho\rangle$
   $+a_5\langle \mathcal{G}_{1,ii}\mathcal{G}_{1,jj}\phi,\rho\rangle + a_6\langle \mathcal{G}_{1,ij}\mathcal{G}_{1,ij}\phi,\rho\rangle + a_7\langle \mathcal{G}_{2,ii}\phi,\rho\rangle + a_8\langle \mathcal{G}_{1,ii}\phi_{;jj},\rho\rangle$
   $+a_9\langle \mathcal{F}_{1,i;i}\phi,\rho\rangle + a_{10}\langle \mathcal{G}_{1,ii}E\phi,\rho\rangle + a_{11}\langle \mathcal{G}_{1,jj;i}\phi_{;i},\rho\rangle + a_{12}\langle \mathcal{G}_{1,ij;j}\phi_{;i},\rho\rangle$
   $+a_{18}\langle \tau\mathcal{G}_{1,ii}\phi,\rho\rangle + a_{19}\rho_{ij}\langle \mathcal{G}_{1,ij}\phi,\rho\rangle + b_2\langle \mathcal{G}_{1,ij}\phi_{;ij},\rho\rangle + b_3\langle \mathcal{E}_1\phi,\rho\rangle$
   $+b_7\langle \mathcal{F}_{1,i}\phi_{;i},\rho\rangle\}dx$
   $+\int_{\partial M}\{a_{13}\langle \mathcal{G}_{1,aa}\phi_{;m},\rho\rangle + a_{14}\langle \mathcal{G}_{1,aa}\phi,\rho_{;m}\rangle + a_{15}L_{bb}\langle \mathcal{G}_{1,aa}\phi,\rho\rangle$
   $+a_{16}\langle \mathcal{G}_{1,aa;m}\phi,\rho\rangle + a_{17}\langle \mathcal{G}_{1,am;a}\phi,\rho\rangle + b_4\langle \mathcal{G}_{1,mm}\phi_{;m},\rho\rangle$
   $+b_5\langle \mathcal{G}_{1,mm}\phi,\rho_{;m}\rangle + b_6\langle \mathcal{G}_{1,mm}L_{aa}\phi,\rho\rangle + b_8\langle \mathcal{F}_{1,m}\phi,\rho\rangle$
   $+b_9\langle \mathcal{G}_{1,am}\phi_{:a},\rho\rangle + b_{10}\langle \mathcal{G}_{1,mm;m}\phi,\rho\rangle + b_{11}L_{ab}\langle \mathcal{G}_{1,ab}\phi,\rho\rangle\}dy.$

As the lack of commutativity in the vector valued case plays no role, we restrict henceforth therefore to the scalar setting. We begin with:

Lemma 2.9.4 *All the constants a_i of Lemma 2.9.3 vanish. We have that $b_8 + b_{11} = 0$ and $b_2 + b_7 = 0$.*

Proof: We adopt the notation of Lemma 2.1.10. Let $(\theta_1, ..., \theta_{m-1})$ be the usual periodic parameters on the torus $\mathbb{T}^{m-1}$. Give $M := \mathbb{T}^{m-1} \times [0,1]$ an initial Riemannian metric of the form

$$ds_M^2 = g_{ab}(\theta,r)d\theta^a \circ d\theta^b + dr \circ dr\,.$$

Let Δ be the scalar Laplacian for this metric. Let

$$\mathfrak{D} = \Delta + \sum_{r=1}^{\infty} (G_r^{ab} \partial_a^\theta \partial_b^\theta + F_r^a \partial_a^\theta) t^r .$$

Since $\partial_\mu^x \partial_\nu^x u = u_{;\mu\nu} + \Gamma_{\mu\nu}{}^\sigma u_{;\sigma}$, when we express $\mathfrak{D}$ in the form given in Equation (2.9.a), we have:

$$\mathcal{G}_r^{ab} = G_r^{ab}, \quad \mathcal{F}_r^a = F_r^a + \Gamma_{bc}{}^a G_r^{bc}, \quad \text{and} \quad \mathcal{F}_r^m = \Gamma_{ab}{}^m G_r^{ab} .$$

Let $\phi = \phi(r)$. By Lemma 2.1.10, $\beta_n(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_n(\phi, \rho, \Delta, \mathcal{B})$ is independent of the structures G_r^{ab} and F_r^a. This implies the vanishing of all the a_i except $\{a_{10}, a_{12}\}$. It also implies $b_8 + b_{11} = 0$ and $b_2 + b_7 = 0$; the presence of the term $\mathcal{F}_r^m$ was missed in [190] and noted subsequently by Park [299].

We show $a_{12} = 0$ by applying the same argument to $M := S^1 \times [0,1]$ where $D_t = -\partial_r^2 - \partial_\theta^2 + t\mathcal{G}_{1,12}(r,\theta)\partial_r\partial_\theta$. Finally, we apply Lemma 2.1.2 to see $a_{10} = a_1 = 0$. $\square$

As $a_1 = 0$, $\beta_2(\phi, \rho, \mathfrak{D}, \mathcal{B}) = \beta_2(\phi, \rho, D_0, \mathcal{B})$. This proves Theorem 2.9.1 (1).

We complete the proof of Theorem 2.9.1 by establishing:

Lemma 2.9.5

1. $b_1 = \frac{1}{2\sqrt{\pi}}$, $b_2 = -\frac{1}{2}$, $b_3 = -\frac{1}{2}$, $b_4 = 0$, $b_5 = \frac{1}{2}$, $b_6 + b_{11} = -\frac{1}{4}$.

2. $b_7 = -\frac{1}{2}$, $b_8 = -\frac{5}{16}$, $b_6 = -\frac{9}{16}$, $b_{11} = \frac{5}{16}$.

3. $b_9 = -\frac{5}{8}$.

4. $b_{10} = \frac{7}{16}$.

Proof: Let $D_t = e^t D_0$. Then by Lemma 2.2.6 and Theorem 2.3.3,

$$\begin{aligned} \beta_3(\phi, \rho, \mathfrak{D}, \mathcal{B}) &= \beta_3(\phi, \rho, D_0, \mathcal{B}) + \tfrac{1}{4}\beta_1(\phi, \rho, D_0, \mathcal{B}) \\ &= \beta_3(\phi, \rho, D_0, \mathcal{B}) - \tfrac{1}{2\sqrt{\pi}} \int_{\partial M} \langle \phi, \rho \rangle dy, \\ \beta_4(\phi, \rho, \mathfrak{D}, \mathcal{B}) &= \beta_4(\phi, \rho, D_0, \mathcal{B}) + \tfrac{1}{2}\beta_2(\phi, \rho, D_0, \mathcal{B}) \\ &= \beta_4(\phi, \rho, D_0, \mathcal{B}) + \int_M \Big\{ \tfrac{1}{2}\langle \phi_{;ii}, \rho \rangle + \tfrac{1}{2}\langle E\phi, \rho \rangle \Big\} dx \\ &\quad + \int_{\partial M} \Big\{ \tfrac{1}{4}\langle L_{aa}, \rho \rangle - \tfrac{1}{2}\langle \phi, \rho_{;m} \rangle \Big\} dy . \end{aligned} \tag{2.9.c}$$

On the other hand, since $\mathcal{G}_1 = -g_0$, $\mathcal{F}_{1,i} = 0$, and $\mathcal{E}_1 = -E$,

$$\begin{aligned} \beta_3(\phi, \rho, \mathfrak{D}, \mathcal{B}) &= \beta_3(\phi, \rho, D_0, \mathcal{B}) - b_1 \int_{\partial M} \langle \phi, \rho \rangle dy, \\ \beta_4(\phi, \rho, \mathfrak{D}, \mathcal{B}) &= \beta_4(\phi, \rho, D_0, \mathcal{B}) + \int_M \Big\{ - b_2 \langle \phi_{;ii}, \rho \rangle - b_3 \langle E\phi, \rho \rangle \Big\} dx \\ &\quad + \int_{\partial M} \Big\{ - b_4 \langle \phi_{;m}, \rho \rangle - b_5 \langle \phi, \rho_{;m} \rangle - (b_6 + b_{11}) L_{aa} \langle \phi, \rho \rangle \Big\} dy . \end{aligned} \tag{2.9.d}$$

The relations of Assertion (1) follow from Displays (2.9.c) and (2.9.d).

To establish Assertion (2) of the Lemma, we apply Lemma 2.2.7 where $M = [0,1] \times S^1$, $f \in C^\infty(M)$, $\Delta = -\partial_r^2 - \partial_\theta^2$, $D_t = \Delta + f$, and

$$\check{D}_t := e^{tf}(\partial_t + \Delta + f)e^{-tf} - \partial_t = \Delta + 2tf_{;i}\partial_i + tf_{;ii} - t^2 f_{;i}^2 .$$

We use Lemma 1.2.1 to see that the endomorphism defined by the operator $\Delta + f$ is, with our sign convention, given by $-f$. Since Dirichlet boundary conditions are unchanged, we may use Lemma 2.2.7 and Theorem 2.3.3 to compute, after integrating by parts, that

$$\begin{aligned}
& \beta_4(\phi,\rho,\check{\mathfrak{D}},\mathcal{B}) = \beta_4(\phi,\rho,\Delta+f,\mathcal{B}) + \beta_2(\phi,f\rho,\Delta+f,\mathcal{B}) \qquad (2.9.e)\\
& \quad +\tfrac{1}{2}\beta_0(\phi,f^2\rho,\Delta+f,\mathcal{B})\\
= \; & \beta_4(\phi,\rho,\Delta,\mathcal{B}) + \int_M \Big\{ \tfrac{1}{2}\langle f\phi,\Delta\rho\rangle + \tfrac{1}{2}\langle\Delta\phi,f\rho\rangle + \tfrac{1}{2}\langle f\phi,f\rho\rangle\\
& \quad -\langle\Delta\phi,f\rho\rangle - \langle f\phi,f\rho\rangle + \tfrac{1}{2}\langle\phi,f^2\rho\rangle \Big\} dx + \int_{\partial M} \Big\{ \tfrac{1}{2}\langle (f\phi)_{;m},\rho\rangle\\
& \quad +\tfrac{1}{2}\langle\phi,(f\rho)_{;m}\rangle - \tfrac{1}{8}f_{;m}\langle\phi,\rho\rangle - \langle\phi,(f\rho)_{;m}\rangle \Big\} dy\\
= \; & \beta_4(\phi,\rho,\Delta,\mathcal{B}) + \int_M \Big\{ \langle -f_{;i}\phi_{;i},\rho\rangle - \tfrac{1}{2}\langle f_{;ii}\phi,\rho\rangle \Big\} dx\\
& \quad -\tfrac{5}{8}\int_{\partial M} \langle f_{;m}\phi,\rho\rangle dy .
\end{aligned}$$

On the other hand, we have $\mathcal{G}_{1,ij} = 0$, $\mathcal{F}_{1;i} = 2f_{;i}$ and $\mathcal{E}_1 = f_{;ii}$. Thus

$$\begin{aligned}
\beta_4(\phi,\rho,\check{\mathfrak{D}},\mathfrak{B}) &= \beta_4(\phi,\rho,\Delta,\mathcal{B}) + \int_M \{ b_3\langle f_{;ii}\phi,\rho\rangle\\
+ \quad & b_7\langle 2f_{;i}\phi_{;i},\rho\rangle \} dx + \int_{\partial M} 2b_8\langle f_{;m}\phi,\rho\rangle dy . \qquad (2.9.f)
\end{aligned}$$

Assertion (2) follows from Equations (2.9.e) and (2.9.f), and from the relations $b_6 + b_{11} = -\frac{1}{4}$ and $b_8 + b_{11} = 0$ derived earlier.

We use Lemma 2.2.8 to establish Assertion (3). Let $M := S^1 \times [0,1]$. Let

$$\begin{aligned}
& D_t := -\partial_r^2 - \partial_\theta^2 - 1 + \varepsilon t\partial_r(\partial_\theta - \sqrt{-1}\mathrm{Id}),\\
& \mathcal{G}_{1,12} = \mathcal{G}_{1,21} = \tfrac{1}{2}\varepsilon \quad \text{and} \quad \mathcal{F}_{1,2} = -\sqrt{-1}\varepsilon .
\end{aligned}$$

Let $\phi(r,\theta) := e^{\sqrt{-1}\theta}\phi_0$ and let $\rho = e^{-\sqrt{-1}\theta}\rho_0$. Then by Lemma 2.2.8, the invariant $\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B})$ is independent of the parameter ε. This implies that

$$0 = \int_M \varepsilon\sqrt{-1}(b_2 - b_7)\langle\partial_r\phi,\rho\rangle dx + \int_{\partial M} (\tfrac{1}{2}b_9 - b_8)\sqrt{-1}\varepsilon\langle\phi,\rho\rangle dy .$$

Thus $b_9 = 2b_8$ and Assertion (3) follows from Assertion (2).

We complete the proof of Theorem 2.9.1 by establishing Assertion (4) of Lemma 2.9.5. We apply Lemma 2.2.9. Let $M = [0,1]$. Let F be a non-negative smooth function on M with $F(0) = 0$ and F vanishing identically near $x = 1$.

Let $\Delta = -\partial_x^2$. Set

$$D_t := -\{(1+tF_x)^{-1}\partial_x\}^2 - F(1+tF_x)^{-1}\partial_x\,.$$

By Theorem 2.3.3 and Lemma 2.2.9, we have

$$0 = \beta_4(1,1,\Delta,\mathcal{B}) = \beta_4(1,1,\mathfrak{D},\mathcal{B}) + \beta_2(1,F_x,\mathfrak{D},\mathcal{B})\,.$$

We compute that

$$\begin{array}{ll} D_t = -\partial_x^2 - F\partial_x + 2tF_x\partial_x^2 + tF_{xx}\partial_x + O(F^2) + O(t^2), & \\ \tilde{D}_0 = -\partial_x^2 + F\partial_x + F_x, & \omega = \frac{1}{2}F, \\ \tilde{\omega} = -\frac{1}{2}F, & E = -\frac{1}{2}F_x + O(F^2), \\ \mathcal{G}_{1,mm} = 2F_x + O(F^2), & \mathcal{F}_{1,m} = F_{xx} + O(F^2)\,. \end{array}$$

Consequently,

$$\begin{aligned} \beta_4(1,1,\mathfrak{D},\mathcal{B}) &= \int_{\partial M}\Big\{\tfrac{1}{2}\langle 1,(\tilde{D}_0 1)_{;m}\rangle + \tfrac{1}{8}\langle E_{;m}1,1\rangle + b_8\langle\mathcal{F}_{1,m}1,1\rangle \\ &\qquad + b_{10}\langle\mathcal{G}_{1,mm;m}1,1\rangle\Big\}dy + O(F^2) \\ \beta_2(1,F_x,\mathfrak{D},\mathcal{B}) &= -\int_{\partial M}\langle 1,F_{xx}\rangle dy + O(F^2)\,. \end{aligned}$$

This yields the relations

$$0 = (\tfrac{1}{2} - \tfrac{1}{16} + b_8 + 2b_{10} - 1)F_{xx};$$

the final assertion follows from this identity and from Assertion (3). □

2.9.2 Proof of Theorem 2.9.2

The interior integrals agree with those computed in Theorem 2.9.1. Thus,

$$\beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) = \beta_n(\phi,\rho,D,\mathcal{B}) \quad \text{for} \quad n = 0,1,2$$

as there are no new boundary integrals. The vanishing arguments used to establish Lemma 2.9.4 can be used to eliminate many terms and to show there exist universal constants so that

$$\beta_3(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_3(\phi,\rho,D_0,\mathcal{B}) + \int_{\partial M}\Big\{c_1\langle\mathcal{G}_{1,mm}\phi,\rho\rangle\Big\}dy, \quad (2.9.g)$$

$$\begin{aligned} &\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_4(\phi,\rho,D_0,\mathcal{B}) \\ &+ \int_M\Big\{-\tfrac{1}{2}\langle\mathcal{G}_{1,ij}\phi_{;ij},\rho\rangle - \tfrac{1}{2}\langle\mathcal{E}_1\phi,\rho\rangle - \tfrac{1}{2}\langle\mathcal{F}_{1,i}\phi_{;i},\rho\rangle\Big\}dx \\ &+ \int_{\partial M}\Big\{c_2\langle\mathcal{G}_{1,mm}\phi_{;m},\rho\rangle + c_3\langle\mathcal{G}_{1,mm}\phi,\rho_{;m}\rangle + c_4\langle\mathcal{G}_{1,mm}L_{aa}\phi,\rho\rangle \\ &\quad + c_5\langle\mathcal{G}_{1,mm}S\phi,\rho\rangle + c_6\langle\mathcal{G}_{1,mm;m}\phi,\rho\rangle + c_7\langle\mathcal{G}_{1,am}\phi_{:a},\rho\rangle \\ &\quad + c_8(\langle\mathcal{F}_{1,m}\phi,\rho\rangle - L_{ab}\langle\mathcal{G}_{1,ab}\phi,\rho\rangle) + c_9\langle S_1\phi,\rho\rangle + c_{10}\langle\Gamma_{a:a}\phi,\rho\rangle \end{aligned}$$

$$+\langle c_{11}\Gamma_a\phi_{:a},\rho\rangle\Big\}dy\,.$$

We complete the proof of Theorem 2.9.2 by evaluating the universal constants of Equation (2.9.g).

Lemma 2.9.6 *We have*

1. $c_1=0$, $c_3=0$, $c_4=0$, $c_6=0$, $c_8=0$, $c_{10}=0$.
2. $c_2=-\frac12$, $c_5=-\frac12$.
3. $c_9=\frac12$.
4. $c_{11}=\frac12$.

Proof: We assume $D_0=\Delta$ and $\mathcal{E}_r=0$ for $r>0$. Let $\phi=1$. We may then apply Lemma 2.2.11 to see β_n is independent of the structures $\{\mathcal{G},\mathcal{F},\Gamma\}$. Since ρ is arbitrary, Assertion (1) follows.

Let $D_t=e^tD_0$ and let $\mathcal{B}_t=\mathcal{B}_0$ be a static family. Then Lemma 2.2.6 and Theorem 2.4.1 imply that

$$\begin{aligned}\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B}_0)&=\beta_4(\phi,\rho,D_0,\mathcal{B}_0)+\tfrac12\beta_2(\phi,\rho,D_0,\mathcal{B}_0)\\&=\beta_4(\phi,\rho,D_0,\mathcal{B}_0)+\int_M\Big\{\tfrac12\langle\phi_{;ii},\rho\rangle+\tfrac12\langle E\phi,\rho\rangle\Big\}dx\qquad(2.9.\text{h})\\&\quad+\int_{\partial M}\tfrac12\langle\mathcal{B}_0\phi,\rho\rangle dy\,.\end{aligned}$$

We have $\mathcal{G}_1=-g_0$, $\mathcal{F}_{1,i}=0$, and $\mathcal{E}_1=-E$. Thus by Equation (2.9.g)

$$\begin{aligned}\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B}_0)&=\beta_4(\phi,\rho,D_0,\mathcal{B}_0)+\int_M\Big\{\tfrac12\langle\phi_{;ii},\rho\rangle+\tfrac12\langle E\phi,\rho\rangle\Big\}dx\\&+\int_{\partial M}\Big\{-c_2\langle\phi_{;m},\rho\rangle-c_5\langle S\phi,\rho\rangle\Big\}dy\,.\qquad(2.9.\text{i})\end{aligned}$$

Assertion (2) now follows from Equations (2.9.h) and (2.9.i).

We modify slightly the argument given to prove Lemma 2.2.7 to establish Assertion (3). Let $M=[0,1]$, let $D_0=-\partial_r^2$, and let $\mathcal{B}_0=\partial_r$. Define

$$\begin{aligned}D_t&:=-\partial_r^2-F+tF_{rr}+t^2F_r^2,\\\mathcal{B}_t&:=\partial_r-tF_r\,.\end{aligned}$$

Let $u(x;t):=e^{tF}$. We check $u=e^{-t\mathfrak{D}_{\mathfrak{B}}}1$ by computing:

$$\begin{aligned}&(\partial_t+D_t)u=(F-tF_{rr}-t^2F_r^2-F+tF_{rr}+t^2F_r^2)e^{tF}=0,\\&\mathcal{B}_tu=(tF_r-tF_r)e^{tF}=0,\\&u|_{t=0}=1\,.\end{aligned}$$

It now follows that

$$\beta(1,1,\mathfrak{D},\mathfrak{B})=\int_Me^{tF}dx\quad\text{so}\quad\beta_4(1,1,\mathfrak{D},\mathfrak{B})=\tfrac12\int_MF^2(x)dx\,.\qquad(2.9.\text{j})$$

Assume ρ vanishes near $r=1$ so only the component $r=0$ where ∂_r is the inward unit normal is relevant. We have

$$\begin{aligned} E &= F, \quad \mathcal{G}_{1,11}=0, \quad \mathcal{F}_{1,1}=0,\\ \mathcal{E}_1 &= F_{rr}, \quad S_1=-F_r\,. \end{aligned}$$

Equation (2.9.g) shows

$$\beta_4(1,1,\mathfrak{D},\mathfrak{B})=\int_M\left\{\tfrac{1}{2}F^2-\tfrac{1}{2}F_{rr}\right\}dx+\int_{\partial M}(-c_9)F_r dy\,. \tag{2.9.k}$$

We use Equation (2.9.j) and Equation (2.9.k) and integrate by parts to see

$$c_9=\tfrac{1}{2}\,.$$

We complete the proof by determining c_{11}. Let $M:=S^1\times[0,1]$. Let

$$D_t:=-\partial_r^2-\partial_\theta^2-2\varepsilon\partial_\theta \quad\text{and}\quad \mathcal{B}_t:=-\partial_r+\delta t(\nabla_{\partial_\theta}-\varepsilon\mathrm{Id}\,)\,.$$

We take $\phi=1$. We have $\omega_\theta=\varepsilon$ so $\nabla_{\partial_\theta}=\partial_\theta+\varepsilon$ and hence $\mathcal{B}_t\phi=0$. Thus $e^{-t\mathfrak{D}_\mathfrak{B}}\phi=1$ so $\beta_4(1,1,\mathfrak{D},\mathfrak{B})=0$. We have $\Gamma=\delta$ and $S_1=-\varepsilon\delta$. Hence

$$0=\int_{\partial M}\delta\varepsilon(c_{11}-c_9)\langle 1,\rho\rangle dy\,.$$

It now follows that $c_{11}=c_9$. $\square$

2.10 Inhomogeneous boundary conditions

We follow the discussion in [49, 51, 50] throughout this section. Let D be an operator of Laplace type. Let $\mathcal{B}$ define a boundary condition. We say that $(D,\mathcal{B})$ is *admissible* if there exists δ with $0<\delta<\frac{\pi}{2}$ so that $(D,\mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}_\delta$. Let ϕ be the initial temperature distribution of the manifold. Let $p=p(x;t)$ be an auxiliary smooth internal heat source and let $\psi=\psi(y;t)$ be the temperature of the boundary. We assume, for the sake of simplicity, that the underlying geometry is fixed. Let $u(x;t)=u_{\phi,p,\psi}(x;t)$ be the subsequent temperature distribution which is defined by the relations

$$\begin{aligned} &(\partial_t+D)u(x;t)=p(x;t) \quad t>0,\\ &\mathcal{B}u(y;t)=\psi(y;t) \quad t>0, y\in\partial M,\\ &u|_{t=0}=\phi\,. \end{aligned} \tag{2.10.a}$$

If $\rho=\rho(x;t)$ is the specific heat of the manifold, then the associated heat content function is given by

$$\beta(\phi,\rho,D,\mathcal{B},p,\psi)(t):=\int_M\langle u_{\phi,p,\psi}(x;t),\rho(x;t)\rangle dx\,.$$

As $t \downarrow 0$, there is a complete asymptotic series

$$\beta(\phi,\rho,D,\mathcal{B},p,\psi)(t) \sim \sum_{n=0}^{\infty} \beta_n(\phi,\rho,D,\mathcal{B},p,\psi)t^{n/2}\,.$$

We set $p = 0$ and $\psi = 0$ to recover the invariants discussed previously

$$\beta_n(\phi,\rho,D,\mathcal{B}) = \beta_n(\phi,\rho,D,\mathcal{B},0,0)\,.$$

The following Theorem permits us to consider the case in which ρ is static.

Theorem 2.10.1 *Let D be an operator of Laplace type. Let $\mathcal{B}$ define a boundary condition so that $(D,\mathcal{B})$ is admissible. Let $\phi = \phi(x)$ be the initial temperature distribution of the manifold. Let $p = p(x;t)$ be an auxiliary smooth internal heat source and let $\psi = \psi(y;t)$ control the temperature of the boundary. Expand the specific heat $\rho(x;t) \sim \sum_{k\ge 0} t^k\rho_k(x)$ in a Taylor series. Then*

$$\beta_n(\phi,\rho,D,\mathcal{B},p,\psi) = \sum_{2k\le n} \beta_{n-2k}(\phi,\rho_k,D,\mathcal{B},p,\psi)\,.$$

Proof: Expand

$$\begin{aligned} \beta(\phi,\rho,D,\mathcal{B},p,\psi)(t) &= \int_M \langle u(x;t),\rho(x;t)\rangle dx \\ \sim \quad & \sum_{k=0}^{\infty} t^k \int_M \langle u(x;t),\rho_k(x)\rangle dx \\ \sim \quad & \sum_{k=0}^{\infty} t^k \beta(\phi,\rho_k,D,\mathcal{B},p,\psi)(t)\,. \end{aligned}$$

The desired conclusion now follows by comparing coefficients of the parameter t in the relevant asymptotic expansions. $\square$

We suppose $\rho = \rho(x)$ is static henceforth. We can use the following Lemma to decouple the invariants:

Lemma 2.10.2 *Let D be an operator of Laplace type. Let $\mathcal{B}$ define a boundary condition so that $(D,\mathcal{B})$ is admissible. Let ϕ be the initial temperature distribution of the manifold. Let $p = p(x;t)$ be an auxiliary smooth internal heat source and let $\psi = \psi(y;t)$ control the temperature of the boundary. Let ρ be the specific heat. Then:*

1. $\beta(\phi,\rho,D,\mathcal{B},p,\psi)(t) = \beta(\phi,\rho,D,\mathcal{B},0,0)(t) + \beta(0,\rho,D,\mathcal{B},p,0)(t)$
   $+\beta(0,\rho,D,\mathcal{B},0,\psi)(t)$.
2. $\beta_n(\phi,\rho,D,\mathcal{B},p,\psi) = \beta_n(\phi,\rho,D,\mathcal{B},0,0) + \beta_n(0,\rho,D,\mathcal{B},p,0)$
   $+\beta_n(0,\rho,D,\mathcal{B},0,\psi)$.

Proof: Since the problem decouples, we have $u_{\phi,p,\psi} = u_{\phi,0,0} + u_{0,p,0} + u_{0,0,\psi}$. The first assertion now follows; the second follows from the first. $\square$

The invariants $\beta_n(\phi,\rho,D,\mathcal{B},0,0) = \beta_n(\phi,\rho,D,\mathcal{B})$ have been studied previously. We therefore focus our attention on expressing the new invariants $\beta_n(0,\rho,D,\mathcal{B},p,0)$ and $\beta_n(0,\rho,D,\mathcal{B},0,\psi)$ in terms of $\beta_n(\cdot,\rho,D,\mathcal{B})$.

For $i, j \geq 0$, we define universal constants by setting

$$c_{ij} := \int_0^1 (1-s)^i s^{j/2} ds \,.$$

Theorem 2.10.3 *Let D be an operator of Laplace type. Let $\mathcal{B}$ define a boundary condition so that $(D, \mathcal{B})$ is admissible. Let $p = p(x;t)$ be an auxiliary smooth internal heat source. Expand $p(x;t) \sim \sum_{k=0}^{\infty} t^k p_k(x)$ in a Taylor series. Then:*

1. $\beta_0(0, \rho, D, \mathcal{B}, p, 0) = 0$.
2. *If $n > 0$, then* $\beta_n(0, \rho, D, \mathcal{B}, p, 0) = \sum_{2i+j+2=n} c_{ij} \beta_j(p_i, \rho, D, \mathcal{B})$.

Proof: As the initial temperature $\phi = 0$, we have $u(0;t) = 0$; the first assertion now follows.

To prove the second assertion, let $U_i := e^{-tD_{\mathcal{B}}} p_i$. Define

$$u_i := \int_0^t (t-s)^i U_i(x;s) ds.$$

It is then immediate that

$$u_i(x;0) = 0 \qquad \text{and} \qquad \mathcal{B}u_i(y;t) = \int_0^t (t-s)^i \mathcal{B}U_i(y;s) ds = 0\,.$$

We now study the corresponding evolution equation:

$$\begin{aligned}
\partial_t u_i &= \{(t-s)^i U_i(x;s)\}|_{s=t} + \int_0^t \partial_t\{(t-s)^i\} U_i(x;s) ds \\
&= \{(t-s)^i U_i(x;s)\}|_{s=t} - \int_0^t \partial_s\{(t-s)^i\} U_i(x;s) ds \\
&= \{(t-s)^i U_i(x;s)\}|_{s=0} + \int_0^t (t-s)^i \partial_s\{U_i(x;s)\} ds \\
&= t^i U_i(x;0) - \int_0^t (t-s)^i D U_i(x;s) ds \\
&= t^i p_i(x) - D u_i(x;t)
\end{aligned}$$

which shows that $(\partial_t + D)U_i = t^i p_i$. It now follows that

$$\begin{aligned}
& \beta(0, \rho, D, \mathcal{B}, p, 0)(t) \\
\sim\ & \sum_{i=0}^{\infty} \beta(0, \rho, D, \mathcal{B}, t^i p_i, 0)(t) \sim \sum_{i=0}^{\infty} \int_M \langle u_i(x;t), \rho \rangle dx \\
\sim\ & \sum_{i=0}^{\infty} \int_M \int_0^t (t-s)^i \langle U_i(x;s), \rho \rangle ds dx \\
\sim\ & \sum_{i=0}^{\infty} \int_0^t (t-s)^i \beta(p_i, \rho, D, \mathcal{B})(s) ds
\end{aligned}$$

$$\sim \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\int_0^t (t-s)^i s^{k/2} ds \cdot \beta_k(p_i,\rho,D,\mathcal{B})$$
$$\sim \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} t^{i+\frac{1}{2}k+1} c_{ik}\beta_k(p_i,\rho,D,\mathcal{B})\,.$$

The second assertion now follows by equating coefficients of t in the asymptotic expansions. □

We apply Theorem 2.10.3 to see

$$\begin{aligned}
&\beta_0(0,\rho,D,\mathcal{B},p,0)=0,\\
&\beta_1(0,\rho,D,\mathcal{B},p,0)=0,\\
&\beta_2(0,\rho,D,\mathcal{B},p,0)=c_{00}\beta_0(p_0,\phi,D,\mathcal{B}),\\
&\beta_3(0,\rho,D,\mathcal{B},p,0)=c_{01}\beta_1(p_0,\phi,D,\mathcal{B}),\\
&\beta_4(0,\rho,D,\mathcal{B},p,0)=c_{02}\beta_2(p_0,\phi,D,\mathcal{B})+c_{10}\beta_0(p_1,\phi,D,\mathcal{B})\,.
\end{aligned}$$

We compute directly that

$$c_{00}=1,\quad c_{01}=\tfrac{2}{3},\quad c_{02}=\tfrac{1}{2},\quad \text{and}\quad c_{10}=\tfrac{1}{2}\,.$$

The following result now follows from Theorems 2.3.3 and 2.4.1:

Theorem 2.10.4 *Let D be an operator of Laplace type. Let $\partial M = C_D \cup C_N$ decompose as a disjoint union of two closed subsets; we permit either C_D or C_N to be empty. Let $\mathcal{B}$ define Dirichlet boundary conditions on C_D and Robin boundary conditions on C_N. Let $p=p(x;t)$ be an auxiliary smooth internal heat source. Expand $p(x;t)\sim\sum_{k=0}^{\infty} t^k p_k(x)$ in a Taylor series. Then:*

1. $\beta_0(0,\rho,D,\mathcal{B},p,0)=0$.
2. $\beta_1(0,\rho,D,\mathcal{B},p,0)=0$.
3. $\beta_2(0,\rho,D,\mathcal{B},p,0)=\int_M\langle p_0,\rho\rangle dx$.
4. $\beta_3(0,\rho,D,\mathcal{B},p,0)=-\frac{4}{3\sqrt{\pi}}\int_{C_D}\langle p_0,\rho\rangle dy$.
5. $\beta_4(0,\rho,D,\mathcal{B},p,0)=\frac{1}{2}\int_M\{\langle p_1,\rho\rangle-\langle D_0p_0,\rho\rangle\}dx$
   $+\frac{1}{2}\int_{C_D}\{\langle\frac{1}{2}L_{aa}p_0,\rho\rangle-\langle p_0,\rho_{;m}\rangle\}dy+\frac{1}{2}\int_{C_N}\langle\mathcal{B}p_0,\rho\rangle dy$.

2.10.2 The heat content asymptotics with a boundary heat pump

We begin our study of the invariants $\beta_n(0,\rho,D,\mathcal{B},0,\psi)$ with two technical results.

Lemma 2.10.5 *Let D be an operator of Laplace type. Let $(D,\mathcal{B})$ be admissible. Assume that $\mathcal{B}$ defines Dirichlet boundary conditions on at least one non-empty component of the boundary. Then given any $\psi\in C^\infty(V|_{\partial M})$, there exists $h\in C^\infty(V)$ so that $Dh=0$ and so that $\mathcal{B}h=\psi$.*

Proof: Let $u(x;t) = u_{0,0,\psi}(x;t)$. One can then take

$$h(x) := \lim_{t\to\infty} u_{0,0,\psi}(x;t);$$

the presence of at least one Dirichlet component eliminates infinite heat buildup and ensures the existence of a harmonic equilibrium solution with the desired boundary values. We also refer to [51] for a probabilistic proof of this assertion. □

The constants $c_{ij} := \int_0^1 (1-s)^i s^{j/2} ds$ enter once again.

Theorem 2.10.6 *Let D be an operator of Laplace type. Let $(D, \mathcal{B})$ be admissible. Assume that $\mathcal{B}$ defines Dirichlet boundary conditions on at least one non-empty component of the boundary. Expand $\psi(y;t) \sim \sum_{k=0}^\infty t^k \psi_k(y)$ in a Taylor series. Choose h_i harmonic so $\mathcal{B}h_i = \psi_i$. Then:*

1. $\beta_0(0, \rho, D, \mathcal{B}, 0, \psi) = 0$.

2. If $n \geq 1$, then $\beta_n(0, \rho, D, \mathcal{B}, 0, \psi) = -\beta_n(h_0, \rho, D, \mathcal{B})$
$\quad - \sum_{2i+j=n, j>0, i>0} i c_{i-1,j} \beta_j(h_i, \rho, D, \mathcal{B})$.

Proof: The first assertion is immediate since $u_{0,0,\psi}(x;0) = 0$. We may assume without loss of generality that $\psi = t^i \psi_i$ in the proof of the second assertion of the Lemma because

$$\beta(0, \rho, D, \mathcal{B}, 0, \psi) \sim \sum_{i=0}^\infty \beta(0, \rho, D, \mathcal{B}, 0, t^i \psi_i)(t)\,.$$

Suppose first that $i = 0$. Let

$$v_0 := e^{-tD_\mathcal{B}} h_0 \quad \text{and} \quad u_0(x;t) = h_0(x) - v_0(x;t)\,.$$

We show $u_0 = u_{0,0,\psi_0}$ by checking the defining relations are satisfied

$$\begin{aligned}
&(\partial_t + D)u_0 = Dh_0 - (\partial_t + D)v_0 = 0,\\
&\mathcal{B}u_0 = \mathcal{B}h_0 - \mathcal{B}v_0 = \psi_0,\\
&u_0(x;0) = h_0(x) - v_0(x;0) = 0\,.
\end{aligned}$$

We then have

$$\begin{aligned}
\beta(0, \rho, D, \mathcal{B}, 0, \psi) &= \int_{\partial M} \langle h_0(x), \rho(x)\rangle dx - \int_{\partial M} \langle v_0(x;t), \rho(x)\rangle dx\\
= &\int_{\partial M} \langle h_0(x), \rho(x)\rangle dx - \beta(h_0, \rho, D, \mathcal{B})(t)\,.
\end{aligned}$$

This implies the desired identity

$$\beta_n(0, \rho, D, \mathcal{B}, 0, \psi) = \begin{cases} 0 & \text{if } n = 0,\\ -\beta_n(h_0, \rho, D, \mathcal{B}) & \text{if } n > 0\,. \end{cases}$$

Next we suppose $i > 0$. Let $v_i := u_{0, it^{i-1}h_i, 0}$ and let $u_i := t^i h_i - v_i$. We show that $u_i = u_{0,0,t^i\psi}$ by checking the defining relations:

$$(\partial_t + D)u_i = it^{i-1}h_i - (\partial_t + D)v_i = it^{i-1}h_i - it^{i-1}h_i = 0,$$

$$\mathcal{B}u_i = t^i\mathcal{B}h_i - \mathcal{B}v_i = t^i\psi_i,$$
$$u_i(x;0) = -v_i(x;0) = 0\,.$$

Therefore

$$\beta(0,\rho,D,\mathcal{B},0,t^i\psi_i)(t) = \int_M \left\{t^i\langle h_i(x),\rho(x)\rangle - \langle v_i(x;t),\rho\rangle\right\}dx$$
$$= \quad t^i\int_M \langle h_i(x),\rho(x)\rangle dx - \beta(0,\rho,D,\mathcal{B},it^{i-1}h_i,0)(t)\,. \qquad \text{(2.10.b)}$$

We use Theorem 2.10.3 to see

$$\beta_n(0,\rho,D,\mathcal{B},it^{i-1}h_i,0) = ic_{i-1,n-2i}\beta_{n-2i}(h_i,\rho,D,\mathcal{B})\,. \qquad \text{(2.10.c)}$$

Let $H_i := \int_M \langle h_i,\rho\rangle dx$. We use Equations (2.10.b) and (2.10.c) to see

$$\beta_n(0,\rho,D,\mathcal{B},0,t^i\psi_i) = \begin{cases} 0 & \text{if } n < 2i \\ -ic_{i-1,0}\beta_0(h_i,\rho,D,\mathcal{B}) + H_i & \text{if } n = 2i, \\ -ic_{i-1,n-2i}\beta_{n-2i}(h_i,\rho,D,\mathcal{B}) & \text{if } n > 2i. \end{cases}$$

Since $ic_{i-1,0} = 1$,

$$\beta_{2i}(0,\rho,D,\mathcal{B},0,t^i\psi_i) = H_i - \int_M \langle h_i(x),\rho(x)\rangle dx = 0\,.$$

This completes the proof of the Theorem. $\square$

Theorem 2.10.6 appears to involve global information since one must first choose h so that $\mathcal{B}h = \psi$. This is, however, not the case. Using the relation

$$0 = Dh = -h_{;mm} - h_{:aa} + L_{aa}h_{;m},$$

Theorem 2.3.3 shows that:

Theorem 2.10.7 *Let D be an operator of Laplace type on a compact Riemannian manifold with smooth boundary ∂M. Let $\mathcal{B}$ define Dirichlet boundary conditions. Let h be harmonic and let $\mathcal{B}h = \psi$. Then*

1. $\beta_1(h,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\psi,\rho\rangle dy$.
2. $\beta_2(h,\rho,D,\mathcal{B}) = \int_{\partial M}\{\langle\frac12 L_{aa}\psi,\rho\rangle - \langle\psi,\rho_{;m}\rangle\}dy$.
3. $\beta_3(h,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\{\frac23\langle\psi,\rho_{;mm}\rangle + \frac13\langle\psi,\rho_{:aa}\rangle$
$+\frac13\langle E\psi,\rho\rangle - \frac23 L_{aa}\langle\psi,\rho_{;m}\rangle + \langle(\frac{1}{12}L_{aa}L_{bb} - \frac16 L_{ab}L_{ab} - \frac16 R_{amma})\psi,\rho\rangle\}dy$.
4. $\beta_4(\psi,\rho,D,\mathcal{B}) = \int_{\partial M}\{\frac12\langle\psi,(\tilde{D}\rho)_{;m}\rangle - \frac14\langle L_{aa}\psi,\tilde{D}\rho\rangle + \langle(\frac18 E_{;m}$
$-\frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac18 L_{ab}L_{ac}L_{bc} - \frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac} + \frac{1}{32}\tau_{;m}$
$+\frac{1}{16}L_{ab:ab})\psi,\rho\rangle - \frac14 L_{ab}\langle\psi_{:a},\rho_{:b}\rangle - \frac18\langle\Omega_{am}\psi_{:a},\rho\rangle + \frac18\langle\Omega_{am}\psi,\rho_{:a}\rangle\}dy$.

Theorem 2.4.1 yields a similar result for Robin boundary conditions:

Theorem 2.10.8 *Let D be an operator of Laplace type on a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}$ define Robin boundary conditions. Let h be harmonic and let $\mathcal{B}h = \psi$.*

1. $\beta_1(h,\rho,D,\mathcal{B})=0$.

2. $\beta_2(h,\rho,D,\mathcal{B})=\int_{\partial M}\langle\psi,\rho\rangle dy$.

3. $\beta_3(h,\rho,D,\mathcal{B})=\frac{4}{3\sqrt{\pi}}\int_{\partial M}\langle\psi,\tilde{\mathcal{B}}\rho\rangle dy$.

4. $\beta_4(h,\rho,D,\mathcal{B})=\int_{\partial M}\{-\frac{1}{2}\langle\psi,\tilde{D}\rho\rangle+\langle(\frac{1}{2}S+\frac{1}{4}L_{aa})\psi,\tilde{\mathcal{B}}\rho\rangle\}dy$.

We use Theorem 2.10.6 to see

$$\begin{aligned}
&\beta_1(0,\rho,D,\mathcal{B},\psi,0)=-\beta_1(h_0,\rho,D,\mathcal{B}),\\
&\beta_2(0,\rho,D,\mathcal{B},\psi,0)=-\beta_2(h_0,\rho,D,\mathcal{B}),\\
&\beta_3(0,\rho,D,\mathcal{B},\psi,0)=-\beta_3(h_0,\rho,D,\mathcal{B})-c_{01}\beta_1(h_1,\rho,D,\mathcal{B}),\\
&\beta_4(0,\rho,D,\mathcal{B},\psi,0)=-\beta_4(h_0,\rho,D,\mathcal{B})-c_{02}\beta_2(h_1,\rho,D,\mathcal{B}).
\end{aligned}$$

Since $c_{01}=\frac{2}{3}$ and $c_{02}=\frac{1}{2}$, the results cited above lead to the following result:

Theorem 2.10.9 *Let D be an operator of Laplace type. Let $\partial M=C_D\cup C_N$ decompose as a disjoint union of two closed subsets; we permit either C_D or C_N to be empty. Let $\mathcal{B}$ define Dirichlet boundary conditions on C_D and Robin boundary conditions on C_N. Let $\psi=\psi(x;t)$ control the heat flow over the boundary. Expand $\psi(y;t)\sim\sum_{n=0}^\infty t^k\psi_k(y)$ in a Taylor series. Then:*

1. $\beta_0(0,\rho,D,\mathcal{B},0,\psi)=0$.

2. $\beta_1(0,\rho,D,\mathcal{B},0,\psi)=\frac{2}{\sqrt{\pi}}\int_{C_D}\langle\psi_0,\rho\rangle dy$.

3. $\beta_2(0,\rho,D,\mathcal{B},0,\psi)=-\int_{C_D}\{\langle\frac{1}{2}L_{aa}\psi_0,\rho\rangle-\langle\psi_0,\rho_{;m}\rangle\}dy-\int_{C_N}\langle\psi_0,\rho\rangle dy$.

4. $\beta_3(0,\rho,D,\mathcal{B},0,\psi)=\frac{2}{\sqrt{\pi}}\int_{C_D}\{\frac{2}{3}\langle\psi_0,\rho_{;mm}\rangle+\frac{1}{3}\langle\psi_0,\rho_{:aa}\rangle$
$+\langle\frac{1}{3}E\psi,\rho\rangle-\frac{2}{3}L_{aa}\langle\psi_0,\rho_{;m}\rangle+\langle(\frac{1}{12}L_{aa}L_{bb}-\frac{1}{6}L_{ab}L_{ab}-\frac{1}{6}R_{amma})\psi_0,\rho\rangle\}dy$
$-\frac{4}{3\sqrt{\pi}}\int_{C_N}\langle\psi_0,\tilde{\mathcal{B}}\rho\rangle dy+\frac{4}{3\sqrt{\pi}}\int_{C_D}\langle\psi_1,\rho\rangle dy$.

5. $\beta_4(0,\rho,D,\mathcal{B},0,\psi)=-\int_{C_D}\{\frac{1}{2}\langle\psi_0,(\tilde{D}\rho)_{;m}\rangle-\frac{1}{4}\langle L_{aa}\psi_0,\tilde{D}\rho\rangle$
$+\langle(\frac{1}{8}E_{;m}-\frac{1}{16}L_{ab}L_{ab}L_{cc}+\frac{1}{8}L_{ab}L_{ac}L_{bc}-\frac{1}{16}R_{ambm}L_{ab}+\frac{1}{16}R_{abcb}L_{ac}$
$+\frac{1}{32}\tau_{;m}+\frac{1}{16}L_{ab:ab})\psi_0,\rho\rangle-\frac{1}{4}L_{ab}\langle\psi_{0:a},\rho_{:b}\rangle-\frac{1}{8}\langle\Omega_{am}\psi_{0:a},\rho\rangle$
$+\frac{1}{8}\langle\Omega_{am}\psi_0,\rho_{:a}\rangle+\frac{1}{4}L_{aa}\langle\psi_1,\rho\rangle-\frac{1}{2}\langle\psi_1,\rho_{;m}\rangle\}dy$
$-\int_{C_N}\{-\frac{1}{2}\langle\psi_0,\tilde{D}\rho\rangle+\langle(\frac{1}{2}S+\frac{1}{4}L_{aa})\psi_0,\tilde{\mathcal{B}}\rho\rangle+\frac{1}{2}\langle\psi_1,\rho\rangle\}dy$.

Proof: If C_D is non-empty, this follows from Theorems 2.10.6, 2.10.7, and 2.10.8. Since we are taking zero initial condition, the heat flows into or out of the manifold across the boundary. Thus, modulo an exponentially small error as $t\downarrow 0$, we may assume that ρ is supported near the boundary. We cut out a small hole in the interior of the manifold to create a new "phantom" Dirichlet component C_0; this again creates an exponentially small error as $t\downarrow 0$ and does not change the heat content asymptotics. Theorem 2.10.6 is now applicable and the desired result follows. □

The discussion above used the fact that $\beta_n(h,\rho,D,\mathcal{B})$ depends only on $\mathcal{B}h$ if h is harmonic. This is a quite general fact.

Theorem 2.10.10 *Let D be an operator of Laplace type on a bundle V and let $\mathcal{B}$ define boundary conditions so that $(D,\mathcal{B})$ is admissible. Let $h \in C^\infty(V)$ be harmonic and let $\psi = \mathcal{B}h \in C^\infty(V|_{\partial M})$. Then there exists a natural partial differential operator $\mathcal{T}_n$ so*

$$\beta_n(h,\rho,D,\mathcal{B}) = \textstyle\int_{\partial M}\langle\psi,\mathcal{T}_n(\rho,D,\mathcal{B})\rangle dy \quad \textit{for} \quad n>0\,.$$

Proof: We follow the discussion in [49]. We apply Lemma 2.1.5. Normalize the interior integrand by setting

$$\beta_n^M(\phi,\rho,D) := \begin{cases} (-1)^k\frac{1}{k!}\int_M\langle D^k\phi,\rho\rangle dx & \text{if } n=2k,\\ 0 & \text{if } n=2k+1\,. \end{cases}$$

We then have

$$\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B}) = \int_{\partial M}\sum_p\langle\mathcal{B}D^p\phi,T_{p,n}\rho\rangle dy$$

for suitably chosen natural partial differential operators $T_{p,n}$. Since h is harmonic, $D^p\phi=0$ for $p>0$. Since $n>0$ the interior integrand vanishes; the boundary integrand vanishes if $p>0$. Consequently

$$\beta_n(h,\rho,D,\mathcal{B}) = \int_{\partial M}\langle\mathcal{B}h,T_{0,n}\rho\rangle dy\,.$$

Setting $\psi := \mathcal{B}h$ and $\mathcal{T}_n(\rho,D,\mathcal{B}) := T_{0,n}\rho$ establishes the desired result. $\square$

2.11 Non-minimal operators

We shall follow the discussion in [188]. Let A and B be positive constants and let E be an endomorphism of the cotangent bundle T^*M. We define

$$D := Ad\delta + B\delta d - E \quad \text{on} \quad C^\infty(\Lambda^1(M))\,.$$

This operator is **not** of Laplace type for $A \neq B$. It is, however, elliptic as was shown in Section 1.6.7. Furthermore, if $\mathcal{B}$ defines absolute or relative boundary conditions, then $(D,\mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$. Let ϕ and ρ be smooth 1 forms; expand $\phi = \phi_i e_i$ and $\rho = \rho_i e_i$.

Theorem 2.11.1 *Let M be a compact Riemannian manifold with smooth boundary ∂M. Let $D = Ad\delta + B\delta d - E$ on $C^\infty(\Lambda^1(M))$. Then:*

1. Let $\mathcal{B}$ define absolute boundary conditions. Then:

(a) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M(\phi,\rho)dx$.

(b) $\beta_1(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\sqrt{A}\int_{\partial M}\phi_m\rho_m dy$.

(c) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\{A(\delta\phi,\delta\rho)+B(d\phi,d\rho)-E(\phi,\rho)\}dx$
$\qquad + \int_{\partial M}A\{-\phi_m\rho_{a:a}-\phi_{a:a}\rho_m-\phi_{m;m}\rho_m-\phi_m\rho_{m;m}+\frac{3}{2}L_{aa}\phi_m\rho_m\}dy$.

2. Let $\mathcal{B}$ define relative boundary conditions. Then

(a) $\beta_0(\phi,\rho,D,\mathcal{B})=\int_M(\phi,\rho)dx$.

(b) $\beta_1(\phi,\rho,D,\mathcal{B})=-\frac{2}{\sqrt{\pi}}\sqrt{B}\int_{\partial M}\phi_a\rho_a dy$.

(c) $\beta_2(\phi,\rho,D,\mathcal{B})=-\int_M\{A(\delta\phi,\delta\rho)+B(d\phi,d\rho)-E(\phi,\rho)\}dx$
$+\int_{\partial M}B\{-\phi_{a:a}\rho_m-\phi_m\rho_{a:a}-\phi_{a;m}\rho_a-\phi_a\rho_{a;m}$
$+L_{ab}\phi_b\rho_a+\frac{1}{2}L_{aa}\phi_b\rho_b\}dy$.

Assertions (1a) and (1b) follow from Lemma 2.1.1. We integrate by parts to replace the interior integrand $-(D\phi_1,\rho_1)$ by the more symmetric integrand

$$-A(d\phi_1,d\rho_1)-B(\delta\phi_1,\delta\rho_0)-(E\phi_1,\rho_1)\,.$$

It then follows there are universal constants $c_i=c_i(A,B)$ so that

$$\beta_1(\phi,\rho,D,\mathcal{B})=\int_{\partial M}\Big\{c_1\phi_a\rho_a+c_2\phi_m\rho_m\Big\}dy,\tag{2.11.a}$$

$$\begin{aligned}\beta_2(\phi,\rho,D,\mathcal{B})&=-\int_M\Big\{A(\delta\phi,\delta\rho)+B(d\phi,d\rho)+(E\phi,\rho)\Big\}dx\\&+\int_{\partial M}\Big\{c_3(\phi_{a;m}\rho_a+\phi_a\rho_{a;m})+c_4(\phi_{a:a}\rho_m+\phi_m\rho_{a:a}\\&+c_5(\phi_{m;m}\rho_m+\phi_m\rho_{m;m})+c_6L_{ab}\phi_a\rho_b+c_7L_{aa}\phi_b\rho_b+c_8L_{aa}\phi_m\rho_m\Big\}dy\,.\end{aligned}$$

As the constants do not involve E, we set $E=0$ henceforth.

2.11.1 Absolute boundary conditions

Let $\mathcal{B}$ define absolute boundary conditions. By Lemma 2.1.16,

$$\begin{aligned}\beta_1(df,\rho,D,\mathcal{B})&=\sqrt{A}\beta_1(df,\rho,\Delta,\mathcal{B}),\quad\text{and}\\\beta_2(df,\rho,D,\mathcal{B})&=A\beta_2(df,\rho,\Delta,\mathcal{B})\,.\end{aligned}$$

The interior integral in Equation (2.11.a) equals the interior integral in the formula of Theorem 2.5.2 for $A\beta_2(df,\rho,D,\mathcal{B})$. Consequently the boundary integrals also agree so, setting $\phi_m=f_{;m}$ and $\phi_a=f_{:a}=f_{;a}$, we have

$$\begin{aligned}&\int_{\partial M}(c_1f_{;a}\rho_a+c_2f_{;m}\rho_m)dy=-\tfrac{2}{\sqrt{\pi}}\sqrt{A}\int_{\partial M}f_{;m}\rho_m dy,\quad\text{and}\\&\int_{\partial M}\Big\{c_3(f_{;am}\rho_a+f_{;a}\rho_{a;m})+c_4(f_{:aa}\rho_m+f_{;m}\rho_{a:a})\\&\qquad+c_5(f_{;mm}\rho_m+f_{;m}\rho_{m;m})+c_6L_{ab}f_{;a}\rho_b\\&\qquad+c_7L_{aa}f_{;b}\rho_b+c_8L_{aa}f_{;m}\rho_m\Big\}dy\\=&\;A\int_{\partial M}\Big\{-f_{:aa}\rho_m-f_{;m}\rho_{a:a}-f_{;mm}\rho_m-f_{;m}\rho_{m;m}+\tfrac{3}{2}L_{aa}f_{;m}\rho_m\Big\}dy\,.\end{aligned}$$

We use this identity to determine the constants c_i. Assertion (1) of Theorem 2.11.1 now follows. □

2.11.2 Relative boundary conditions

Let $\mathcal{B}$ define relative boundary conditions. Let $\phi := -\delta\psi$ where $\psi_{ij} = -\psi_{ji}$ is an anti-symmetric 2 tensor. By Lemma 2.1.16,

$$\beta_1(\delta\psi, \rho, D, \mathcal{B}) = \sqrt{B}\beta_1(\delta\psi, \rho, \Delta, \mathcal{B}), \quad \text{and}$$
$$\beta_2(\delta\psi, \rho, D, \mathcal{B}) = B\beta_2(\delta\psi, \rho, \Delta, \mathcal{B}) .$$

Since $-\delta\psi_i = \psi_{ij;j}$, one has

$$\begin{aligned}
&\int_{\partial M}(c_1\psi_{aj;j}\rho_a + c_2\psi_{mj;j}\rho_m)dy = -\tfrac{2}{\sqrt{\pi}}\sqrt{B}\int_{\partial M}\psi_{aj;j}\rho_a dy, \quad \text{and}\\
&\int_{\partial M}\Big\{c_3(\psi_{aj;jm}\rho_a + \psi_{aj;j}\rho_{a;m}) + c_4(\psi_{aj;j:a}\rho_m + \psi_{mj;j}\rho_{a:a})\\
&\quad +c_5(\psi_{mj;jm}\rho_m + \psi_{mj;j}\rho_{m;m}) + c_6L_{ab}\psi_{aj;j}\rho_b\\
&\quad +c_7L_{aa}\psi_{bj;j}\rho_b + c_8L_{aa}\psi_{mj;j}\rho_m\Big\}dy\\
= \; & B\int_{\partial M}\Big\{-\psi_{aj;j:a}\rho_m - \psi_{mj;j}\rho_{a:a} - \psi_{aj;jm}\rho_a - \psi_{aj;j}\rho_{a;m}\\
&\quad +L_{ab}\psi_{aj;j}\rho_b + \tfrac{1}{2}L_{aa}\psi_{bj;j}\rho_b\Big\}dy .
\end{aligned}$$

Again, the coefficients c_i are determined and Theorem 2.11.1 (2c) follows. □

2.12 Spectral boundary conditions

We follow the discussion in [197, 198] for the material of this section. Let M be a compact Riemannian manifold with smooth boundary. Let γ give a vector bundle V over M a Clifford module structure and let ∇ be a compatible connection on V. The dual connection $\tilde{\nabla}$ on the dual bundle V^* is then compatible with respect to the dual Clifford module structure $\tilde{\gamma}$. Let

$$P := \gamma_i\nabla_{e_i} + \psi_P$$

be an operator of Dirac type on V and let $D := P^2$ be the associated operator of Laplace type. Note that the connection defined by D will not in general be compatible with the Clifford module structure. Let

$$A := -\gamma_m\gamma_a\nabla_{e_a} + \psi_A$$

be an auxiliary tangential operator of Dirac type on $V|_{\partial M}$. As in Section 1.6.6, we suppose that $\ker(A) = \{0\}$ and that A is self-adjoint with respect to some fiber metric on $V|_{\partial M}$. We let Π_A^+ be spectral projection on the span of the positive eigenspaces of A and let

$$\mathcal{B}_A := \Pi_A^+ \oplus \Pi_A^+ P$$

be the associated boundary operator for D. By Lemma 1.6.7, (P, Π_A^+) is elliptic with respect to the cone $\mathcal{K}$ and thus $(D, \mathcal{B}_A)$ is elliptic with respect to the cone $\mathcal{C}$.

Let $\tilde{P}$ be the dual operator of Laplace type on V^*; by Lemma 1.6.7,

$$\tilde{P} = -\tilde{\gamma}_i \nabla_{e_i} + \tilde{\psi}_P$$

and the dual boundary condition is $\Pi_{A\#}$ where

$$\begin{aligned} A^{\#} &= \tilde{\gamma}_m \tilde{A} \tilde{\gamma}_m = -\tilde{\gamma}_m \tilde{\gamma}_a \tilde{\nabla}_{e_a} + \psi_{A\#} \quad \text{and} \\ \psi_{A\#} &= \tilde{\gamma}_m \tilde{\psi}_A \tilde{\gamma}_m + L_{aa}\mathrm{Id}\ . \end{aligned} \tag{2.12.a}$$

Set

$$\beta(\phi, \rho, D, \mathcal{B}_A)(t) := \int_M \langle e^{-tD_{\mathcal{B}_A}} \phi(x;t), \rho(x) \rangle dx\,.$$

We shall **assume** the existence of an appropriate asymptotic series as $t \downarrow 0$

$$\beta(\phi, \rho, D, \mathcal{B}_A)(t) \sim \sum_{n=0}^{\infty} \beta_n(\phi, \rho, D, \mathcal{B}_A) t^{n/2}\,. \tag{2.12.b}$$

Theorem 2.12.1 *Let P be an operator of Dirac type on a compact Riemannian manifold. Let A be admissible with respect to P.*

1. $\beta_0(\phi, \rho, D, \mathcal{B}_A) = \int_M \langle \phi, \rho \rangle dx$.
2. $\beta_1(\phi, \rho, D, \mathcal{B}_A) = -\frac{2}{\sqrt{\pi}} \int_{\partial M} \langle \Pi_A^+ \phi, \Pi_{A\#}^+ \rho \rangle dy$.
3. $\beta_2(\phi, \rho, D, \mathcal{B}_A) = -\int_M \langle D\phi, \rho \rangle dx + \int_{\partial M} \{ -\langle \gamma_m \Pi_A^+ P\phi, \rho \rangle - \langle \gamma_m \Pi_A^+ \phi, \tilde{P}\rho \rangle$
$\quad + \frac{1}{2} \langle (L_{aa} + A + \tilde{A}^{\#} - \gamma_m \psi_P + \psi_P \gamma_m - \psi_A - \tilde{\psi}_A^{\#}) \Pi_A^+ \phi, \Pi_{A\#}^+ \rho \rangle \} dy$.

The first assertion is immediate; the remainder of this section is devoted to the proof of the second and third assertions.

2.12.1 The proof of Theorem 2.12.1 (2)

By Lemma 2.1.1, there is no interior integrand in β_1. If $\mathcal{B}_A \phi = 0$, then $\beta_1(\phi, \rho, D, \mathcal{B}_A) = 0$ by Lemma 2.1.4. Dually by Lemma 2.1.3,

$$\beta_1(\phi, \rho, D, \mathcal{B}_A) = \beta_1(\rho, \phi, \tilde{D}, \mathcal{B}_{A\#}) = 0 \text{ if } \mathcal{B}_{A\#}\rho = 0\,.$$

Since the boundary integrand must be homogeneous of weight 0, only the pseudo-differential projections Π_A^+ and $\Pi_{A\#}^+$ can appear in the formula for β_1. As a result, there exist universal constants so

$$\begin{aligned} &\beta_1(\phi, \rho, D, \mathcal{B}_A) \\ &= \int_{\partial M} \left\{ c_0(m) \langle \Pi_A^+ \phi, \Pi_{A\#}^+ \rho \rangle + c_1(m) \langle \gamma_m \Pi_A^+ \phi, \Pi_{A\#}^+ \rho \rangle \right\} dy\,. \end{aligned} \tag{2.12.c}$$

By Lemma 2.2.15, $\beta_1(\phi,\rho,P^2,\mathcal{B}_A)=\beta_1(\phi,\rho,(-P)^2,\mathcal{B}_A)$. Replacing P by $-P$ replaces γ_m by $-\gamma_m$. Thus the universal constant $c_1(m)$ vanishes. We can also deduce directly that this invariant is not present using Equation (2.12.f) given below.

We apply Lemma 2.2.16 to the following example to evaluate the constant $c_0(m)$; as we shall use this example subsequently in our study of β_2, we present the notation in a consistent framework.

Example 2.12.2 Let $\{\gamma_1,...,\gamma_m\}$ be skew-adjoint $\ell\times\ell$ matrices satisfying the Clifford commutation relations

$$\gamma_i\gamma_j+\gamma_j\gamma_i=-2\delta_{ij}\mathrm{Id}$$

for $1\le i,j\le m$. Let $M:=\mathbb{T}^{m-1}\times[0,1]$ be given the flat product metric. Let $\varepsilon>0$ and $\delta>0$ be real parameters. Let γ_0 be a self-adjoint idempotent $\ell\times\ell$ matrix which anti-commutes with γ_m. Let $V:=M\times\mathbb{C}^\ell$ and let P be the operator of Dirac type on V given by

$$P:=\gamma_m\partial_r+\sum_{a=1}^{m-1}\gamma_a\partial_a^\theta+\varepsilon\gamma_m\gamma_0\,.$$

We take into account the fact that ∂_r is the inward unit normal when $r=0$ and the outward unit normal when $r=1$ to define an operator of Dirac type on $V|_{\partial M}$ which anti-commutes with γ_m by setting

$$A:=\left\{\begin{array}{lll}-\gamma_m\gamma_a\partial_a^\theta+\delta\gamma_0 & \text{if} & r=0.\\ \gamma_m\gamma_a\partial_a^\theta+\delta\gamma_0 & \text{if} & r=1\,.\end{array}\right.$$

We have $D=(\Delta_M+\varepsilon^2)\otimes\mathrm{Id}$ and $A=(\Delta_{\mathbb{T}^{m-1}}+\delta_0^2)\otimes\mathrm{Id}$. Consequently $\ker(A)=\{0\}$ so A is admissible with respect to P. Let $\mathcal{B}_A$ be the associated boundary condition. Let $M_0:=[0,1]$ and let $V_0:=[0,1]\times\mathbb{C}^\ell$. Let

$$\begin{aligned}&P_0:=\gamma_m\partial_r+\varepsilon\gamma_m\gamma_0 \text{ on } V_0,\\ &D_0=(\gamma_m\partial_r+\varepsilon\gamma_m\gamma_0)^2=(-\partial_r^2+\varepsilon^2)\otimes\mathrm{Id} \text{ on } V_0\,.\end{aligned}$$

Let Π_0 and Π_1 be orthogonal projection on the $+1$ and -1 eigenspace of γ_0, respectively. As γ_0 anti-commutes with γ_m,

$$\gamma_m\Pi_0=\Pi_1\gamma_m \quad\text{and}\quad \gamma_m\Pi_1=\Pi_0\gamma_m\,.$$

Let $\mathcal{B}_0$ define mixed boundary conditions for D_0 where

$$\mathcal{B}_0\phi:=\Pi_0\phi|_{\partial M}\quad\oplus\quad\Pi_1(\partial_r+\varepsilon\gamma_0)\phi|_{\partial M}\,. \tag{2.12.d}$$

Let $\phi=\phi(r)$ and $\rho=\rho(r)$. We suppose ρ vanishes identically near $r=1$ so only the component of ∂M, where $r=0$ and where ∂_r is the inward unit normal, is relevant. By Lemma 2.2.16

$$\beta_n(\phi,\rho,D,\mathcal{B}_A)=(2\pi)^{m-1}\beta_n(\phi,\rho,D_0,\mathcal{B}_0)\,.$$

By Theorem 2.5.1,

$$(2\pi)^{m-1}\beta_1(\phi,\rho,D_0,\mathcal{B}_0) = -\tfrac{2}{\sqrt{\pi}}\int_{\partial M}\langle\Pi_0\phi,\tilde{\Pi}_0\rho\rangle dy\,.$$

Since $\Pi_A^+\phi = \Pi_0\phi$ and $\Pi_{A\#}^+\rho = \tilde{\Pi}_0\rho$, Equation (2.12.c) implies

$$\beta_1(\phi,\rho,D,\mathcal{B}) = c_0(m)\int_{\partial M}\langle\Pi_0\phi,\tilde{\Pi}_0\rho\rangle dy\,.$$

Equating $\beta_1(\phi,\rho,D_0,\mathcal{B}_0)$ and $\beta_1(\phi,\rho,D,\mathcal{B})$ then establishes the second assertion of Theorem 2.12.1 by showing

$$c_0(m) = -\tfrac{2}{\sqrt{\pi}}\,.$$

Remark: The constant $c_0(m)$ was determined in [197] using a special case computation and the present calculation should be regarded as providing a useful compatibility check on that calculation.

2.12.2 The proof of Theorem 2.12.1 (3)

We begin by expressing β_2 in terms of invariants with undetermined universal coefficients. The following Ansatz is far from rigorous but represents a reasonable formulation of what we believe the analysis will show the invariants to be.

Ansatz 2.12.3 *There exist universal constants c_i so that*

$$\begin{aligned}\beta_2(\phi,\rho,D,\mathcal{B}_A) = -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\Big\{&-\langle\gamma_m\Pi_A^+P\phi,\rho\rangle\\ -\langle\gamma_m\Pi_A^+\phi,\tilde{P}\rho\rangle + \langle(c_2(A+\tilde{A}^\#) &+ c_3L_{aa} + c_4(\gamma_m\psi_P - \psi_P\gamma_m)\\ +c_5(\psi_A+\tilde{\psi}_A^\#))\Pi_A^+\phi,\Pi_{A\#}\rho\rangle\Big\}dy\,.\end{aligned}$$

Justification: We use Lemma 2.1.1 to see that the interior integral for β_2 is given by $-\langle D\phi,\rho\rangle$. We define the normalized invariant $\mathcal{C}$, which is given by a suitable boundary integral, by the identity

$$\begin{aligned}\beta_2(\phi,\rho,D,\mathcal{B}_A) &= \mathcal{C}(\phi,\rho,D,\mathcal{B}_A) - \int_M\langle D\phi,\rho\rangle dx\\ &\quad+\int_{\partial M}\Big\{-\langle\gamma_m\Pi_A^+P\phi,\rho\rangle - \langle\gamma_m\Pi_A^+\phi,\tilde{P}\rho\rangle\Big\}dy\,.\end{aligned}$$

As we must replace γ_m by $-\tilde{\gamma}_m$ in passing to the dual structures, we have

$$\begin{aligned}0 &= \beta_2(\phi,\rho,D,\mathcal{B}_A) - \beta_2(\rho,\phi,\tilde{D},\mathcal{B}_{A\#})\\ &= \mathcal{C}(\phi,\rho,D,\mathcal{B}_A) - \mathcal{C}(\rho,\phi,\tilde{D},\mathcal{B}_{A\#}) - \int_M\Big\{\langle D\phi,\rho\rangle - \langle\phi,\tilde{D}\rho\rangle\Big\}dx\\ &\quad+\int_{\partial M}\Big\{-\langle\gamma_m\Pi_A^+P\phi,\rho\rangle - \langle\gamma_m\Pi_A^+\phi,\tilde{P}\rho\rangle - \langle\phi,\tilde{\gamma}_m\Pi_{A\#}^+\tilde{P}\rho\rangle\end{aligned}$$

$$-\langle P\phi,\tilde{\gamma}_m\Pi^+_{A\#}\rho\rangle\Big\}dy\,.$$

We now use the Green's formula given in Lemma 1.6.7 to see that

$$\mathcal{C}(\phi,\rho,D,\mathcal{B}_A)=\mathcal{C}(\rho,\phi,\tilde{D},\mathcal{B}_{A\#})\,. \tag{2.12.e}$$

Thus, if need be, after averaging over the $\mathbb{Z}_2$ symmetry, which interchanges the roles of ϕ and of ρ, we can assume that the integral expressions for $\mathcal{C}$ are symmetric in the roles of ϕ and ρ. This symmetry, of course, motivated the normalizations in the definition of $\mathcal{C}$ given above.

If $\mathcal{B}_A\phi=0$, then $\mathcal{C}(\phi,\rho,D,\mathcal{B}_A)=0$ by Lemma 2.1.4. Therefore by Equation (2.12.e) we also have $\mathcal{C}(\phi,\rho,D,\mathcal{B}_A)=0$ if $\mathcal{B}_{A\#}\rho=0$. We now argue as when discussing oblique boundary conditions in Section 2.8, after eliminating divergence terms, that the integral formula for $\mathcal{C}$ is divisible by expressions which are bilinear in multiples and tangential covariant derivatives of

$$\{\Pi^+_A\phi,\Pi^+_AP\phi\}\qquad\text{and}\qquad\{\Pi^+_{A\#}\rho,\Pi^+_{A\#}\tilde{P}\rho\}\,;$$

see the discussion in Lemma 2.1.5 concerning this point.

Since the boundary integrals defining $\mathcal{C}$ have total weight 1, terms which are bilinear in $\Pi^+_AP\phi$ and $\Pi^+_{A\#}\tilde{P}\rho$ do not appear. By Equation (1.6.k),

$$\gamma_m\Pi^+_A=(\mathrm{Id}\,-\tilde{\Pi}^+_{A\#})\gamma_m\quad\text{so}\quad\int_{\partial M}\langle\gamma_m\Pi^+_A\phi,\Pi^+_{A\#}\rho\rangle dy=0\,. \tag{2.12.f}$$

We use Equation (2.12.f) to see that terms which are bilinear in $\Pi^+_AP\phi$ and $\Pi^+_{A\#}\rho$ or in $\Pi^+_A\phi$ and $\Pi^+_{A\#}P\rho$ do not involve γ_m. Taking into account the symmetry of Equation (2.12.e), we see that these terms would have the form

$$\int_{\partial M}b_0(\langle\Pi^+_AP\phi,\Pi^+_{A\#}\rho\rangle+\langle\Pi^+_A\phi,\Pi^+_{A\#}\tilde{P}\rho\rangle)dy\,.$$

We use the invariance of β_2 under the $\mathbb{Z}_2$ symmetry which replaces P by $-P$ discussed in Lemma 2.2.15 to see $b_0=0$.

Consequently there exists a natural first order tangential differential operator $\mathcal{C}_1$ and a natural endomorphism $\mathcal{C}_2$ so that

$$\mathcal{C}(\phi,\rho,D,\mathcal{B}_A)=\int_{\partial M}\langle(\mathcal{C}_1+\mathcal{C}_2)\Pi^+_A\phi,\Pi^+_{A\#}\rho\rangle dy\,.$$

The natural first order tangential differential operators of total weight 1 which seem to enter in this setting are the four operators

$$\{A,\gamma_mA,A\gamma_m,\gamma_mA\gamma_m\}\,.$$

Thus we can express

$$\begin{aligned}&\mathcal{C}_1(\phi,\rho,D,\mathcal{B}_A)\\ =\;&\int_{\partial M}\langle(b_1A+b_2\gamma_mA+b_3A\gamma_m+b_4\gamma_mA\gamma_m)\Pi^+_A\phi,\Pi^+_{A\#}\rho\rangle dy\,.\end{aligned}$$

Implicit in this ansatz is the remark that invariants $\gamma_a\psi_P\gamma_a$, $\gamma_a\psi_A\gamma_a$, and so forth do not appear at this level. Such invariants would violate Lemma 2.2.17.

Since $\tilde{\gamma}_m \tilde{A} \tilde{\gamma}_m = A^\#$, Equation (2.12.e) implies $b_1 = b_4$. Furthermore, we replace P by $-P$ to replace γ_m by $-\gamma_m$ and use Lemma 2.2.15 to see that we have $b_2 = b_3 = 0$. Set $b_1 = b_4 = c_2$. This leads to the invariant

$$c_2 \int_{\partial M} \langle (A + \tilde{A}^\#)\Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle dy \,.$$

The only metric term of total weight 1 is L_{aa}; we denote the coefficient of this term by c_3; by Lemma 2.2.15, $\gamma_m L_{aa}$ does not enter. This leads to

$$c_3 \int_{\partial M} L_{aa} \langle \Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle dy \,.$$

The remaining terms exist in flat space. Terms which are linear in ψ_P are

$$\mathcal{P}(D, A) := b_5 \psi_P + b_6 \gamma_m \psi_P + b_7 \psi_P \gamma_m + b_8 \gamma_m \psi_P \gamma_m \,.$$

Interchanging the roles of V and V^* replaces ψ_P by $\tilde{\psi}_P$ and γ_m by $-\tilde{\gamma}_m$. Thus

$$\mathcal{P}(\tilde{D}, A^\#) = b_5 \tilde{\psi}_P - b_6 \tilde{\gamma}_m \tilde{\psi}_P - b_7 \tilde{\psi}_P \tilde{\gamma}_m + b_8 \tilde{\gamma}_m \tilde{\psi}_P \tilde{\gamma}_m \,.$$

Setting $\tilde{\mathcal{P}}(\tilde{D}, A^\#) = P(D, A)$ then yields the relations $b_6 + b_7 = 0$. We use Lemma 2.2.15 to see $b_5 = b_8 = 0$. Setting $b_6 = -b_7 = c_4$ gives rise to

$$c_4 \int_{\partial M} \langle (\gamma_m \psi_P - \psi_P \gamma_m)\Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle dy \,.$$

Terms which are linear in ψ_A have the form

$$\mathcal{Q}(D, A) := b_9 \psi_A + b_{10} \gamma_m \psi_A + b_{11} \psi_A \gamma_m + b_{12} \gamma_m \psi_A \gamma_m \,.$$

Again, in flat space, interchanging A and $A^\#$ replaces ψ_A by $\tilde{\gamma}_m \tilde{\psi}_A \tilde{\gamma}_m$. This yields the relations $b_9 = b_{12}$; we use Lemma 2.2.15 to see $b_{10} = b_{11} = 0$. Setting $c_5 = b_9 = b_{12}$ gives rise to the invariant

$$c_5 \int_{\partial M} \langle (\psi_A + \tilde{\psi}_A^\#)\Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle dy \,.$$

We complete the proof by using Lemma 2.2.17 to see the coefficients c_i are independent both of the dimension of the underlying manifold and also of the rank of the bundle in question. □

We evaluate the constants of Ansatz 2.12.3 to establish Theorem 2.12.1 (3).

Lemma 2.12.4

1. $c_2 = \frac{1}{2}$, $c_4 = -\frac{1}{2}$, $c_5 = -\frac{1}{2}$.

2. $c_3 = \frac{1}{2}$.

Proof: We adopt the notation discussed in Example 2.12.2. The flat connection is compatible with the Clifford module structure. It is not, however, the only possible compatible connection. Let ϱ_a be auxiliary real constants. We define a compatible connection by setting $\omega_a := \varrho_a \mathrm{Id}$. We shall take $\phi = \phi(r)$ and $\rho = \rho(r)$. We suppose ρ vanishes identically near $r = 1$ so only the boundary component $r=0$ is relevant. Let

$$\begin{aligned} P: &= \gamma_m\partial_r + \gamma_a\partial_a^\theta + \varepsilon\gamma_m\gamma_0 = \gamma_m\nabla_{\partial_r} + \gamma_a\nabla_{\partial_a^\theta} + \varepsilon\gamma_m\gamma_0 - \varrho_a\gamma_a, \\ A: &= -\gamma_m\gamma_a\partial_a^\theta + \delta\gamma_0 = -\gamma_m\gamma_a\nabla_{\partial_a^\theta} + \delta\gamma_0 + \gamma_m\gamma_a\varrho_a\,. \end{aligned}$$

Since γ_a and γ_0 anti-commute with γ_m,

$$\begin{aligned} &\psi_P = \varepsilon\gamma_m\gamma_0 - \varrho_a\gamma_a \quad \text{so} \quad \gamma_m\psi_P - \psi_P\gamma_m = -2\varepsilon\gamma_0 - 2\gamma_m\gamma_a\varrho_a, \\ &\psi_A = \delta\gamma_0 + \gamma_m\gamma_a\varrho_a \quad \text{so} \quad \psi_A + \gamma_m\psi_A\gamma_m = 2\delta\gamma_0 + 2\gamma_m\gamma_a\varrho_a\,. \end{aligned}$$

We compute:

$$\begin{aligned} &\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M \langle D\phi,\rho\rangle dx + \int_{\partial M}\Big\{ -\langle\gamma_m\Pi_0\gamma_m(\partial_r+\varepsilon\gamma_0)\phi,\rho\rangle \\ &\qquad +\langle\gamma_m\Pi_0\phi,\tilde\gamma_m(\partial_r+\varepsilon\tilde\gamma_0)\rho\rangle + \langle((2c_2+2c_5)\delta\gamma_0 \\ &\qquad +(2c_5-2c_4)\gamma_m\gamma_a\varrho_a - 2c_4\varepsilon\gamma_0)\Pi_0\phi,\tilde\Pi_0\rho\rangle\Big\}dy \\ = &\ -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\Big\{\langle\Pi_1(\partial_r+\varepsilon\gamma_0)\phi,\rho\rangle - \langle\Pi_0\phi,\partial_r\rho\rangle \\ &\qquad +\langle[(-1-2c_4)\varepsilon\gamma_0 + (2c_2+2c_5)\delta\gamma_0 \\ &\qquad +(2c_5-2c_4)\gamma_m\gamma_a\varrho_a]\Pi_0\phi,\rho\rangle\Big\}dy\,. \end{aligned}$$

On the other hand, since $P_0^2 = -(\partial_r^2-\varepsilon^2)\mathrm{Id}$, the connection defined by D_0 is the trivial connection. Thus by Theorem 2.5.1,

$$\begin{aligned} &(2\pi)^{m-1}\beta_2(\phi,\rho,D_0,\mathcal{B}_0) = -(2\pi)^{m-1}\int_{M_0}\langle D_0\phi,\rho\rangle dr \\ + &\quad (2\pi)^{m-1}\int_{\partial M_0}\Big\{\langle\phi_{+;m}+S\phi_+,\rho_+\rangle - \langle\phi_-,\rho_{-;m}\rangle\Big\}dy_0 \\ = &\quad -\int_M\langle D\phi,\rho\rangle dx + \int_{\partial M}\Big\{\langle\Pi_1(\partial_r+\varepsilon\gamma_0)\phi,\rho\rangle - \langle\Pi_0\phi,\partial_r\rho\rangle\Big\}dy\,. \end{aligned}$$

We have $\gamma_0\Pi_0 = \Pi_0$. Set $\varrho_1 = 1$, $\varrho_a = 0$ for $a>0$, and $\gamma_0 = \sqrt{-1}\gamma_m\gamma_1$. We may then equate $\beta_2(\phi,\rho,D,\mathcal{B})$ with $(2\pi)^{m-1}\beta_2(\phi,\rho,D_0,\mathcal{B}_0)$ to complete the proof of Assertion (1) by deriving the relations

$$2c_2+2c_5=0, \quad 2c_5-2c_4=0, \quad \text{and} \quad 2c_4=-1\,.$$

To study the coefficient of L_{aa} we consider a variant of Example 2.12.2. Let $f=f(r)$ be a smooth function with $f(0)=0$ and $f(1)=0$. Instead of taking a flat metric on $M:=\mathbb{T}^{m-1}\times[0,1]$, we take the warped product metric

$$ds_M^2 = dr^2 + e^{2f}(d\theta_1^2+...+d\theta_{m-1}^2)\,.$$

The volume element is then given by $dx = g dr d\theta_1...d\theta_{m-1}$ where $g:=e^{(m-1)f}$. Let $\{\Theta^1,...,.\Theta^m\}$ be skew-adjoint matrices satisfying the Clifford commutation relations

$$\Theta^i\Theta^j + \Theta^j\Theta^i = -2\delta^{ij} \quad \text{for} \quad 0\le i,j\le m\,.$$

We set $\gamma_m := \Theta^m$, $\gamma_a := e^f\Theta^a$, and $\gamma^a := e^{-f}\Theta^a$. This defines a Clifford module structure on $M \times \mathbb{C}^\ell$. Let γ_0 be an auxiliary idempotent self-adjoint matrix which anti-commutes with γ_m. We define

$$P := \gamma_m\partial_r + \gamma^a\partial_a^\theta \quad \text{and} \quad A := -\gamma_m\gamma^a\partial_a^\theta + \delta\gamma_0\,.$$

Let $\phi = \phi(r)$ and $\rho = \rho(r)$. We suppose ρ vanishes identically near $r = 1$. We may then apply Lemma 2.2.16 and Theorem 2.5.1 to compute:

$$\begin{aligned} &\beta_2(\phi, g^{-1}\rho, D, \mathcal{B}_A) = (2\pi)^{m-1}\beta_2(\phi, \rho, D_0, \mathcal{B}_0) \\ = &\int_M -\langle D_0\phi, \rho\rangle dx + \int_{\partial M}\Big\{\langle\partial_r\phi_+, \rho_+\rangle - \langle\phi_-, \partial_r\rho_-\rangle\Big\}dy\,. \end{aligned}$$

Since $\tilde{P}g^{-1}\rho = -g^{-1}\tilde{\gamma}_m\partial_r\rho$, we may conclude

$$\begin{aligned} 0 = \int_{\partial M}&\langle(c_2(A + A^\#) + c_3L_{aa} + c_4(\gamma_m\psi_P - \psi_P\gamma_m) \\ &+c_5(\psi_A + \tilde{\psi}_A^\#))\Pi_0\phi, \Pi_0\rho\rangle dy\,. \end{aligned} \tag{2.12.g}$$

Motivated by Lemma 1.1.7, we define $\omega_m = 0$ and $\omega_a = \frac{1}{2}\partial_r f\cdot\gamma_m\gamma_a$. Then

$$\begin{aligned} &\Gamma_{mab} = \Gamma_{amb} = -\Gamma_{abm} = \partial_r f\cdot e^{2f}\delta_{ab}, \\ &\Gamma_{ma}{}^b = \Gamma_{am}{}^b = \partial_r f\delta_{ab}, \\ &\Gamma_{ab}{}^m = -\partial_r f\cdot e^{2f}\delta_{ab}, \quad \text{so} \\ &\gamma_{m;m} = 0, \\ &\gamma_{a;m} = \partial_r f\cdot\gamma_a - \Gamma_{ma}{}^b\gamma_b = 0, \\ &\gamma_{m;a} = -\Gamma_{am}{}^b\gamma_b + [\omega_a, \gamma_m] = -\partial_r f\cdot\gamma_a + \tfrac{1}{2}\partial_r f[\gamma_m\gamma_a, \gamma_m] \\ &\qquad = -\partial_r\cdot\gamma_a + \partial_r f\cdot\gamma_a = 0, \\ &\gamma_{a;b} = -\Gamma_{ba}{}^m\gamma_m + [\omega_b, \gamma_a] \\ &\qquad = \partial_r f\cdot e^{2f}\gamma_m\delta_{ab} + \tfrac{1}{2}\partial_r f[\gamma_m\gamma_b, \gamma_a] \\ &\qquad = \partial_r f\cdot e^{2f}\gamma_m\delta_{ab} - \partial_r f\cdot e^{2f}\gamma_m\delta_{ab} = 0\,. \end{aligned}$$

Consequently $\psi_P = -\gamma^a\omega_a = -\frac{1}{2}\partial_r f\gamma^a\gamma_m\gamma_a = -\frac{1}{2}(m-1)\partial_r f\gamma_m$ and thus $\gamma_m\psi_P - \psi_P\gamma_m = 0$. Similarly

$$\begin{aligned} &\psi_A = \gamma_m\gamma^a\omega_a + \delta\gamma_0 = -\tfrac{1}{2}(m-1)\partial_r f + \delta\gamma_0, \quad \text{so} \\ &\psi_A + \tilde{\psi}_{A^\#} = L_{aa}\mathrm{Id} + 2\delta\gamma_0\,. \end{aligned}$$

Thus Equation (2.12.g) implies $0 = \int_{\partial M}(c_3 + c_5)L_{aa}\langle\Pi_0\phi, \Pi_0\rho\rangle dy$ and thus $c_3 + c_5 = 0$. Consequently $c_3 = -c_5 = \frac{1}{2}$. This completes the proof of Lemma 2.12.4 and thereby the proof of Theorem 2.12.1. $\square$

Chapter 3

Heat Trace Asymptotics

3.0 Introduction

In Chapter 3, we discuss the heat trace asymptotics. As in Chapter 2, let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M and let V be a smooth vector bundle over M. We shall introduce a number of different constants c_i; we clear the notation at the beginning of each new section. Thus, for example, the constants of Section 3.3 have no relation to the constants of Section 3.4.

We say that $(D, \mathcal{B})$ is *admissible* on V over M if $\mathcal{B}$ defines boundary conditions for an operator D of Laplace type on $C^\infty(V)$ and if $(D, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}_\delta$ for some $0 \le \delta < \frac{\pi}{2}$. Note that in contrast Chapter 2, we do not impose an ellipticity condition for the dual structures on V^*. If it is not necessary to specify the manifold and vector bundle in question, we shall simply say $(D, \mathcal{B})$ is admissible. If the boundary of M is empty, then the boundary condition is irrelevant and we shall say that D is admissible.

Let $D_\mathcal{B}$ be the associated realization where (D, B) is admissible. We apply Theorem 1.4.5. The fundamental solution of the heat equation, $e^{-tD_\mathcal{B}}$, is characterized by Display (1.4.d). This operator is of trace class and can be represented by a smooth kernel function $K(t, x, x_1, D, \mathcal{B}) : V_{x_1} \to V_x$ so that

$$e^{-tD_\mathcal{B}}\phi(x;t) = \int_M K(t, x, x_1, D, \mathcal{B})\phi(x_1)dx_1 \,.$$

Let $F \in C^\infty(\operatorname{End}(V))$ be an auxiliary smooth endomorphism of V, which is used for localization. There is a complete asymptotic expansion as $t \downarrow 0$

$$\operatorname{Tr}_{L^2}\{Fe^{-tD_\mathcal{B}}\} \sim \sum_{n=0}^{\infty} a_n(F, D, \mathcal{B})t^{(n-m)/2}$$

where the *heat trace coefficients* a_n are locally computable. If $\mathcal{B}$ defines spectral boundary conditions, an expansion of this form exists only for $0 \le n \le m-1$.

If f is a scalar function, then we shall set

$$a_n(f, D, \mathcal{B}) := a_n(f \cdot \mathrm{Id}, D, \mathcal{B}) .$$

Let ∇ be the natural connection defined by D; we refer to Lemma 1.2.1 for details. ∇_{e_m} denote covariant differentiation with respect to the inward unit normal vector field. There are local endomorphism valued invariants $e_n(x, D)$ and $e_{n,k}(y, D, \mathcal{B})$ which are defined on M and on ∂M, respectively, so that

$$\begin{aligned} a_n(F, D, \mathcal{B}) &= \int_M \mathrm{Tr}_{V_x}\Big\{F(x)e_n(x, D)\Big\}dx \\ &+ \sum_{k=0}^{n-1}\int_{\partial M}\mathrm{Tr}_{V_y}\Big\{\nabla_{e_m}^k F(y)\cdot e_{n,k}(y, D, \mathcal{B})\Big\}dy . \end{aligned} \tag{3.0.a}$$

The interior invariants e_n vanish for n odd; the boundary invariants are generically non-zero for $n \geq 1$.

The study of the asymptotic coefficients a_n will comprise the focus of Chapter 3. In Sections 3.1 and 3.2, we will discuss the functorial properties of these invariants. Some of these properties are analogous to properties discussed previously for the heat content asymptotics while some of the properties are new.

In Section 3.3, we discuss these invariants for closed manifolds; this is a non-trivial computation as there is no analogue of Assertions (2) and (3) of Theorem 1.3.12 giving a general formula for the interior invariants.

In Sections 3.4 and 3.5, we study the boundary integrals for Dirichlet and Robin boundary conditions, respectively. The invariants a_n for $n \leq 3$ where D is the scalar Laplacian were originally determined by Kennedy, Critchley, and Dowker [256]. We shall follow the functorial method outlined in [84] to study these invariants. We also refer to [270, 285, 287, 344, 345] for related work. In Section 3.6, we extend these results to study mixed boundary conditions.

In Section 3.7, we present a few geometrical applications of these formulae to relate the spectrum of the Laplacian to the geometry of the underlying manifold. We begin by presenting the Berger-Tanno result [58, 347] that standard spheres in dimension at most 6 are characterized by the spectrum of the scalar Laplacian. If instead of considering just the scalar Laplacian, one considers the p form valued Laplacian, then more information is available. We present results of Patodi [301] for closed manifolds and of Park [297, 298] for manifolds with boundary showing that certain geometric properties are determined by the spectrum of the p form valued Laplacians for $p = 0, 1, 2$. We refer to [35] for a more complete bibliography of some papers in the field of spectral geometry.

Section 3.8 deals with the supertrace asymptotics for the Witten Laplacian. In Section 3.9, we study the large and small energy limits of the heat trace asymptotics.

In Section 3.10, we study the heat trace asymptotics for transmission boundary conditions and in Section 3.11, we study the heat trace asymptotics for transfer boundary conditions. In Section 3.12, we discuss time-dependent phenomena. As was the case with the heat content asymptotics, a bit of care must be taken in defining the relevant kernel functions.

In Section 3.13 we study heat trace asymptotics which are related to the eta function and use these results to study spectral boundary conditions in Section 3.14.

We conclude in Sections 3.15, 3.16, and 3.17 by studying the heat trace asymptotics defined by operators that are not of Laplace type. Section 3.15 deals with second order non-minimal operators, i.e. with operators which have non-scalar leading symbol. In Section 3.16 we study fourth order operators, and in Section 3.17 we study pseudo-differential operators.

There is a lengthy history involved in the study of such asymptotic expansions. Weyl [360] began the study in 1915 by determining the a_0 term which lead to the asymptotic formula

$$\lambda_n \sim n^{2/m}$$

for the Laplacian on a compact manifold. Minakshisundaram and Pleijel [281, 282, 283, 310, 312] then pioneered the study of other terms in the asymptotic expansion. Work of Kac [250], McKean and Singer [278], and Patodi [301, 302] was extremely influential.

We refer to the Bibliography for a small selection of additional references on the subject. Additional references may be found, among other locations, in P. Bérard and M. Berger [35] and in the Bibliography by H. Schroeder contained in [189].

3.1 Functorial properties I

The heat trace asymptotics, like the heat content asymptotics, have a number of functorial properties. Some of these properties are similar to properties of the heat content asymptotics. For example, Lemma 2.1.2 for the heat content asymptotics and Lemma 3.1.1 for the heat trace asymptotics relate to shifting the spectrum by adding a scalar multiple of the identity. Similarly, Lemma 2.1.6 for the heat content asymptotics and Lemma 3.1.2 for the heat trace asymptotics involve Fourier expansions when $D_{\mathcal{B}}$ is self-adjoint.

There are properties of the heat content asymptotics that have no corresponding analogues for the heat trace asymptotics. Lemma 2.1.1 which gives the interior terms for the heat content asymptotics and Lemma 2.1.3 which gives a duality relationship are such properties.

On the other hand, there are properties of the heat trace asymptotics not mirrored by the heat content asymptotics. For example, the variational formulae of Section 3.1.10 have no analogue for the heat content asymptotics. Thus, although there are some formal similarities between the heat content asymptotics and the heat trace asymptotics, each setting has to be treated differently.

In this section, we discuss properties which are common to many boundary conditions. We postpone until the next section a discussion of transmission, transfer, and spectral boundary conditions. We also postpone a discussion of time-dependent phenomena until that time.

3.1.1 Shifting the spectrum

The following property of the heat trace asymptotics is a direct generalization of the corresponding property given in Lemma 2.1.2 for the heat content asymptotics.

Lemma 3.1.1 *Let $(D,\mathcal{B})$ be admissible, let F be a smooth endomorphism of V, and let ε be an auxiliary real parameter. Then*

$$a_n(F,D-\varepsilon\mathrm{Id}\,,\mathcal{B})=\textstyle\sum_{2k\le n}\frac{\varepsilon^k}{k!}a_{n-2k}(F,D,\mathcal{B})\,.$$

Proof: Let $u:=e^{-tD_{\mathcal{B}}}\phi$, let $D_\varepsilon:=D-\varepsilon\mathrm{Id}\,$, and let $u_\varepsilon=e^{t\varepsilon}u$. In the proof of Lemma 2.1.2, we showed that $u_\varepsilon=e^{-tD_{\varepsilon,\mathcal{B}}}\phi$. This implies

$$K(t,x,x_1,D_\varepsilon,\mathcal{B})=e^{t\varepsilon}K(t,x,x_1,D,\mathcal{B})\,.$$

Consequently

$$\begin{aligned}
&\mathrm{Tr}_{\,L^2}\left\{Fe^{-tD_{\varepsilon,\mathcal{B}}}\right\}\\
=\;&\int_M\mathrm{Tr}_{\,V_x}\left\{F(x)K(t,x,x,D_\varepsilon,\mathcal{B})\right\}dx\\
=\;&e^{t\varepsilon}\int_M\mathrm{Tr}_{\,V_x}\left\{F(x)K(t,x,x,D,\mathcal{B})\right\}dx\\
=\;&e^{t\varepsilon}\mathrm{Tr}_{\,L^2}\left\{Fe^{-tD_{\mathcal{B}}}\right\}.
\end{aligned}$$

We may now establish the Lemma by equating powers of t in the asymptotic expansions. $\square$

3.1.2 Heat trace asymptotics for self-adjoint operators

Let V be equipped with a positive definite Hermitian inner product. Assume that $(D,\mathcal{B})$ is admissible and that $D_{\mathcal{B}}$ is self-adjoint. Let $\{\phi_i,\lambda_i\}$ be the discrete spectral resolution of $D_{\mathcal{B}}$, which was discussed previously in Theorem 1.4.18. If $\rho_i\in C^\infty(V)$, define $\rho_1(x)\otimes\rho_2^*(x_1)\in\mathrm{Hom}\,(V_{x_1},V_x)$ by setting

$$\{\rho_1(x)\otimes\rho_2^*(x_1)\}(\cdot)=(\cdot,\rho_2(x_1))\rho_1(x)\,.$$

The following result generalizes Lemma 2.1.6 to the current setting.

Lemma 3.1.2 *Let $(D,\mathcal{B})$ be admissible and self-adjoint. Let $\{\phi_i,\lambda_i\}$ be the associated discrete spectral resolution. Then:*

1. $K(t,x,x_1,D,\mathcal{B})=\sum_i e^{-t\lambda_i}\phi_i(x)\otimes\phi_i^*(x_1)$.

2. $\operatorname{Tr}_{L^2}\{Fe^{-tD_\mathcal{B}}\} = \sum_i e^{-t\lambda_i} \int_M \operatorname{Tr}_{V_x}\{F(x)(\phi_i(x) \otimes \phi_i^*(x))\}dx$.

3. Let $f \in C^\infty(M)$. *Then* $\operatorname{Tr}_{L^2}\{fe^{-tD_\mathcal{B}}\} = \sum_i e^{-t\lambda_i} \int_M f(x)|\phi_i(x)|^2 dx$.

4. $Tr_{L^2}\{e^{-tD_\mathcal{B}}\} = \sum_i e^{-t\lambda_i}$.

Proof: If $\phi \in C^\infty(V)$, then the associated Fourier coefficients are given by

$$\sigma_i(\phi) := \int_M (\phi, \phi_i)(x)dx\,.$$

We showed in the proof of Lemma 2.1.6 that

$$\begin{aligned} e^{-tD_\mathcal{B}}\phi &= \sum_i e^{-t\lambda_i}\sigma_i(\phi)\phi_i = \sum_i e^{-t\lambda_i}\left\{\int_M (\phi,\phi_i)(x_1)dx_1\right\}\phi_i(x) \\ &= \int_M \left\{\sum_i e^{-t\lambda_i}\phi_i(x)\otimes\phi_i^*(x_1)\right\}\phi(x_1)dx_1\,. \end{aligned}$$

The first assertion of the Lemma now follows as a kernel function in this form is unique. The second assertion of the Lemma is an immediate consequence of the first. The third assertion follows from the second assertion since

$$\operatorname{Tr}_{V_x}\{f(\phi_i(x)\otimes\phi_i^*(x))\} = f(x)|\phi_i(x)|^2\,.$$

The final assertion follows by setting $f = 1$ since $\int_M |\phi_i|^2 dx = 1$. $\square$

3.1.3 Heat trace asymptotics in the 1 *dimensional setting*

The following result gives the heat trace asymptotics for the Laplacian on the circle S^1 and the interval $[0,\pi]$; it will play an important role in the subsequent determination of certain normalizing constants.

Lemma 3.1.3

1. Let $\Delta_{S^1} := -\partial_\theta^2$ *on* S^1. *Then:*

  (a) $a_0(1, \Delta_{S^1}) = \sqrt{\pi}$.

  (b) $a_n(1, \Delta_{S^1}) = 0$ *for* $n > 0$.

2. Let $\Delta_{[0,\pi]} = -\partial_x^2$ *on* $[0,\pi]$, *let* $\mathcal{B}_D$ *define Dirichlet boundary conditions, and let* $\mathcal{B}_N$ *define pure Neumann boundary conditions. Then:*

  (a) $a_0(1, \Delta_{[0,\pi]}, \mathcal{B}_D) = \frac{1}{2}\sqrt{\pi}$ *and* $a_0(1, \Delta_{[0,\pi]}, \mathcal{B}_N) = \frac{1}{2}\sqrt{\pi}$.

  (b) $a_1(1, \Delta_{[0,\pi]}, \mathcal{B}_D) = -\frac{1}{2}$ *and* $a_1(1, \Delta_{[0,\pi]}, \mathcal{B}_N) = +\frac{1}{2}$.

  (c) $a_n(1, \Delta_{[0,\pi]}, \mathcal{B}_D) = 0$ *and* $a_n(1, \Delta_{[0,\pi]}, \mathcal{B}_N) = 0$ *for* $n \geq 2$.

Proof: We use Example 1.5.12, Example 1.5.14, and Lemma 3.1.2 (4) to see

$$\operatorname{Spec}\{\Delta_{S^1}\} = \left\{n^2\right\}_{n=-\infty}^{\infty} \quad \text{so} \quad \operatorname{Tr}_{L^2}\left\{e^{-t\Delta_{S^1}}\right\} = \sum_{n=-\infty}^{\infty} e^{-tn^2},$$

$$\operatorname{Spec}\{\Delta_{[0,\pi],\mathcal{B}_D}\} = \left\{n^2\right\}_{n=1}^{\infty} \quad \text{so} \quad \operatorname{Tr}_{L^2}\left\{e^{-t\Delta_{[0,\pi],\mathcal{B}_D}}\right\} = \sum_{n=1}^{\infty} e^{-tn^2},$$

$$\text{Spec}\{\Delta_{[0,\pi],\mathcal{B}_N}\} = \left\{n^2\right\}_{n=0}^{\infty} \quad \text{so} \quad \text{Tr}_{L^2}\left\{e^{-t\Delta_{[0,\pi],\mathcal{B}_N}}\right\} = \sum_{n=0}^{\infty} e^{-tn^2}.$$

We apply the analysis used to prove Lemma 2.3.5 to establish Assertion (1a) by computing

$$\begin{aligned} a_0(1,\Delta_{S^1}) &= \lim_{t\downarrow 0}\left\{\sqrt{t}\cdot\text{Tr}_{L^2}e^{-t\Delta_{[0,\pi]}}\right\} = \lim_{t\downarrow 0}\left\{\sqrt{t}\cdot\sum_{n=-\infty}^{\infty} e^{-tn^2}\right\} \\ &= \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}\,. \end{aligned}$$

The structures on S^1 are flat and hence all the derivatives of the symbol of Δ vanish. Assertion (1b) now follows since the local formulae for a_n are homogeneous of weight n in the derivatives of the structures defining Δ, as we shall see presently in Section 3.1.8; this also follows from standard arguments using Gaussian sums as

$$\lim_{t\downarrow 0}\left\{\sqrt{t}\cdot\text{Tr}_{L^2}e^{-t\Delta}\right\} = \sqrt{\pi} + O(t^k) \text{ for any } k\in\mathbb{N}.$$

We use the computations performed above to see that

$$\begin{aligned} \text{Tr}_{L^2}\{e^{-t\Delta_{[0,\pi],\mathcal{B}_D}}\} &= \tfrac{1}{2}\text{Tr}_{L^2}\{e^{-t\Delta_{S^1}}\} - \tfrac{1}{2}, \\ \text{Tr}_{L^2}\{e^{-t\Delta_{[0,\pi],\mathcal{B}_N}}\} &= \tfrac{1}{2}\text{Tr}_{L^2}\{e^{-t\Delta_{S^1}}\} + \tfrac{1}{2}\,. \end{aligned}$$

Consequently,

$$\begin{aligned} a_n(1,\Delta,\mathcal{B}_D) &= \begin{cases} \frac{1}{2}a_n(1,\Delta_{S^1}) & \text{if } n\neq 1, \\ \frac{1}{2}a_1(1,\Delta_{S^1}) - \frac{1}{2} & \text{if } n = 1, \end{cases} \\ a_n(1,\Delta,\mathcal{B}_N) &= \begin{cases} \frac{1}{2}a_n(1,\Delta_{S^1}) & \text{if } n\neq 1, \\ \frac{1}{2}a_1(1,\Delta_{S^1}) + \frac{1}{2} & \text{if } n = 1\,. \end{cases} \end{aligned}$$

Assertion (2) now follows from Assertion (1). $\square$

3.1.4 Conditions that imply the heat trace coefficients are real

The following observation of Branson and Gilkey [84] will be useful in eliminating certain invariants from consideration.

Lemma 3.1.4 *Let $(D,\mathcal{B})$ be admissible.*

1. *If V, D, $\mathcal{B}$, and F are real, then $a_n(F,D,\mathcal{B})$ is real.*
2. *If $D_{\mathcal{B}}$ and F are self-adjoint, then $a_n(F,D,\mathcal{B})$ is real.*

Proof: Let V, D, $\mathcal{B}$, and ϕ be real. Let $u := e^{-tD_{\mathcal{B}}}\phi$. Then $\bar{u}$ also satisfies the defining relations of Display (1.4.d) so $u = \bar{u}$ and u is real. It now follows that $\bar{K} = K$ and hence K is real. Assertion (1) now follows.

If $D_{\mathcal{B}}$ is self-adjoint, then the eigenvalues λ_i are real. Let ϕ_i be the associated eigensections. We then have that

$$((\phi_i\otimes\phi_i^*)v_1, v_2) = (\phi_i, v_2)(v_1,\phi_i)\,.$$

Modulo taking the complex conjugate, this is symmetric in the roles of v_1 and v_2. This shows that $\phi_i \otimes \phi_i^*$ is self-adjoint. Thus

$$\begin{aligned} & \overline{\mathrm{Tr}}_{V_x}\{F(x)(\phi_i(x) \otimes \phi_i^*(x))\} = \mathrm{Tr}_{V_x}\{(\phi_i(x) \otimes \phi_i^*(x))^* F^*(x)\} \\ = \; & \mathrm{Tr}_{V_x}\{(\phi_i(x) \otimes \phi_i^*(x))F(x)\} = \mathrm{Tr}_{V_x}\{F(x)\phi_i(x) \otimes \phi_i^*(x)\}\,. \end{aligned}$$

This shows that the fiber trace is real. Assertion (2) now follows from Lemma 3.1.2. □

3.1.5 Direct sums

As was the case for the heat content asymptotics, the heat trace asymptotics are additive with respect to direct sums.

Lemma 3.1.5 *Let $(D_1, \mathcal{B}_1)$ be admissible on V_1 over M and let $(D_2, \mathcal{B}_2)$ be admissible on V_2 over M. Let*

$$D := D_1 \oplus D_2 \quad \textit{and} \quad \mathcal{B} := \mathcal{B}_1 \oplus \mathcal{B}_2\,.$$

Then $(D, \mathcal{B})$ is admissible. Let $F_i \in \mathrm{End}\,(V_i)$ and let $F := F_1 \oplus F_2$. Then

1. *$K(t, x, x_1, D, \mathcal{B}) = K(t, x, x_1, D_1, \mathcal{B}_1) \oplus K(t, x, x_1, D_2, \mathcal{B}_2)$.*
2. *$a_n(F, D, \mathcal{B}) = a_n(F_1, D_1, \mathcal{B}_1) + a_n(F_2, D_2, \mathcal{B}_2)$.*

Proof: Let $u_i := e^{-tD_{i,\mathcal{B}_i}}\phi_i$ and let $u := e^{-tD_\mathcal{B}}\phi$. We showed in the proof of Lemma 2.1.7 that we may express $u = u_1 \oplus u_2$. The first assertion now follows; the second follows from the first by equating terms in the associated asymptotic expansions. □

3.1.6 Product Formulas

The following Lemma generalizes Lemma 2.1.8.

Lemma 3.1.6 *For $i = 1, 2$, let $(D_i, \mathcal{B}_i)$ be admissible on vector bundles V_i over compact Riemannian manifolds (M_i, g_i). Let M_1 be closed so no boundary condition is needed for D_1. Let $(M, g) := (M_1, g_1) \times (M_2, g_2)$ be the product Riemannian manifold and let $V := \pi_1^* V_1 \otimes \pi_2^* V_2$ be the tensor product bundle over M. Let $F := F_1 \otimes F_2$ for $F_i \in \mathrm{End}\,(V_i)$. Define D and $\mathcal{B}$ by setting*

$$D := D_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes D_2 \quad \textit{and} \quad \mathcal{B} := \mathrm{Id}_1 \otimes \mathcal{B}_2 \quad \textit{on} \quad C^\infty(V)\,.$$

We have that $(D, \mathcal{B})$ is admissible and:

1. *$K(t, (x_1, x_2), (\bar{x}_1, \bar{x}_2), D, \mathcal{B}) = K(t, x_1, \bar{x}_1, D_1) \otimes K(t, x_2, \bar{x}_2, D_2, \mathcal{B}_2)$.*
2. *$a_n(F, D, \mathcal{B}) = \sum_{n_1+n_2=n} a_{n_1}(F_1, D_1) a_{n_2}(F_2, D_2, \mathcal{B}_2)$.*

Proof: Let $\phi_i \in C^\infty(V_i)$ over M_i, let $\phi := \phi_1 \otimes \phi_2 \in C^\infty(V)$ over M. Let $u := e^{-tD_\mathcal{B}}\phi$, $u_1 := e^{-tD_{1,\mathcal{B}_1}}\phi_1$, and $u_2 := e^{-tD_2}\phi_2$. We showed in the proof of Lemma 2.1.8 that $u = u_1 \otimes u_2$. Assertion (1) now follows. Assertion (2) follows from Assertion (1). □

We can use Lemma 3.1.6 to extend Lemma 3.1.3 to the higher dimensional setting. The following result is now immediate:

Lemma 3.1.7 *Give $\mathbb{T}^m := S^1 \times ... \times S^1$ the flat product metric. Then*

$$a_0(1, \Delta_{\mathbb{T}^m}) = \pi^{m/2} \quad \textit{and} \quad a_n(1, \Delta_{\mathbb{T}^m}) = 0 \textit{ for } n > 0\,.$$

Remark 3.1.8 We require that $\mathcal{B}_1$ is a partial differential operator to ensure that $\mathcal{B}_1 \otimes \mathrm{Id}$ is again an operator within the context with which we are working; if $\mathcal{B}_1$ defined spectral boundary conditions, then $\mathcal{B}_1$ would be a pseudo-differential operator over M_1. However, $\mathcal{B}_1 \otimes \mathrm{Id}$ would not be a pseudo-differential operator over M and hence would not define admissible boundary conditions. As we shall see presently, Lemma 3.1.6 implies that, with the exception of spectral boundary conditions, if we express the local integrands for the heat trace asymptotics in terms of a Weyl basis, then the coefficients are, up to a multiplicative normalizing constant, universal. This normalizing constant will then be determined using Lemma 3.1.7. This is not, however, the case for spectral boundary conditions as we shall see in Section 3.14; the coefficients there exhibit highly non-trivial dependence on m.

3.1.7 Dimensional analysis

Theorem 2.1.12 extends to this context.

Theorem 3.1.9 *Let $(D, \mathcal{B})$ be admissible and let $c > 0$. Then:*

1. $K(t, x, x_1, c^2D, \mathcal{B}) = c^m K(c^2t, x, x_1, D, \mathcal{B})$.

2. $a_n(F, c^2D, \mathcal{B}) = c^{n-m} a_n(F, D, \mathcal{B})$.

3. $e_n(x, c^2D) = c^n e_n(x, D)$.

4. $e_{n,k}(y, c^2D, \mathcal{B}) = c^{n-1-k} e_{n,k}(y, D, \mathcal{B})$.

Proof: Let $u(x;t) := e^{-tD_{\mathcal{B}}}\phi$, let $u_c(x;t) := u(x;c^2t)$, and let $D_c := c^2D$. The argument used to prove Lemma 2.1.13 then shows

$$u_c = e^{-tD_{c,\mathcal{B}}}\phi\,.$$

Consequently after taking into account the fact that $dx_c = c^{-m}dx$, we see

$$\begin{aligned} u_c(x;t) = \int_M K(t, x, x_1, c^2D, \mathcal{B})\phi(x_1)d_cx_1 \\ = \quad c^{-m}\int_M K(t, x, x_1, c^2D, \mathcal{B})\phi(x_1)dx_1 \\ = \quad u(x;c^2t) = \int_M K(c^2t, x, x_1, D, \mathcal{B})\phi(x_1)dx_1\,. \end{aligned}$$

Assertion (1) now follows. We use Assertion (1) to see

$$\begin{aligned} \mathrm{Tr}_{L^2}(Fe^{-tD_{c,\mathcal{B}}}) = \int_M \mathrm{Tr}_{V_x}\Big\{F(x)K(t, x, x, D_c, \mathcal{B})\Big\}d_cx \\ = \int_M \mathrm{Tr}_{V_x}\Big\{F(x)K(c^2t, x, x, D, \mathcal{B})\Big\}dx = \mathrm{Tr}_{L^2}(Fe^{-(c^2t)D_{\mathcal{B}}})\,. \end{aligned}$$

Equating terms in the asymptotic expansions proves Assertion (2).

The third and fourth assertions follow from the second after taking into account change in the volume form and the normal vector field. Alternatively, one can also appeal directly to the Seeley-Greiner calculus [224, 341]. □

3.1.8 Expressing the invariants e_n and $e_{n,k}$ relative to a Weyl basis

We now give a second proof of Lemma 1.7.7 which is not based on a detailed analysis of the Seeley calculus and which uses only the fact that the invariants e_n and $e_{n,k}$ are locally computable.

As in Section 2.2.4, we let indices i, j, k, l range from 1 through m and index a local orthonormal frame $\{e_1, ..., e_m\}$ for TM. Let ∇ be the connection determined by an operator D of Laplace type on V. We let ";" denote the components of multiple covariant differentiation of tensors of all types with respect to the connection ∇ and the Levi-Civita connection of M. Let R_{ijkl} be the components of the curvature tensor of the Levi-Civita connection of M, let Ω_{ij} be the components of the curvature tensor of the connection defined by D, and let E be the endomorphism defined by D. We defined previously in Section 1.7.3

$$\text{weight}\,(R_{ijkl}) = 2, \quad \text{weight}\,(\Omega_{ij}) = 2, \quad \text{and} \quad \text{weight}\,(E) = 2\,.$$

We increase the weight by 1 for every additional explicit covariant derivative which appears. Applying Theorem 3.1.9 and the rescaling arguments discussed in Section 2.2.4 then shows that $e_n(x, D)$ is homogeneous of total weight n in the variables

$$\{R_{ijkl;...},\ E_{;...},\ \Omega_{ij;...}\}\,.$$

Let τ be the scalar curvature and let ρ be the Ricci curvature,

$$\tau := R_{ijji} \quad \text{and} \quad \rho_{ij} := R_{ikkj}\,.$$

The norms of these tensors are given by

$$|\rho|^2 = \rho_{ij}\rho_{ij} = R_{ikkj}R_{illj} \quad \text{and} \quad |R|^2 = R_{ijkl}R_{ijkl}\,.$$

After taking into account the additional invariants which can be constructed from E and from Ω, we obtain the following generalization of Lemma 1.7.5.

Lemma 3.1.10 *Let $\mathcal{E}_{n,m}$ be the space of endomorphism valued invariants of weight n in the derivatives of the symbol of an operator of Laplace type which are defined on the interior of M.*

1. $\mathcal{E}_{0,m} = \text{Span}\,\{\text{Id}\}$.
2. $\mathcal{E}_{2,m} = \text{Span}\,\{\tau\text{Id}\,,\ E\}$.
3. $\mathcal{E}_{4,m} = \text{Span}\,\{\tau_{;kk}\text{Id}\,,\ \tau^2\text{Id}\,,\ |\rho|^2\text{Id}\,,\ |R|^2\text{Id}\,,\ \tau E,\ E^2,\ E_{;kk},\ \Omega_{ij}\Omega_{ij}\}$.

Near the boundary, we assume the frame is chosen so e_m is the inward unit geodesic normal vector field. We let indices a, b, c range from 1 through $m-1$ and index the induced orthonormal frame $\{e_1, ..., e_{m-1}\}$ for $T(\partial M)$. Let L_{ab}

be the second fundamental form. A similar analysis establishes the following generalization of Lemma 1.7.6:

Lemma 3.1.11 *Let $\mathcal{E}^{bd}_{n,m}$ be the space of endomorphism valued invariants of weight n in the derivatives of the symbol of an operator of Laplace type which are defined on the boundary of M.*

1. $\mathcal{E}^{bd}_{0,m} = \mathrm{Span}\,\{\mathrm{Id}\,\}$.
2. $\mathcal{E}^{bd}_{1,m} = \mathrm{Span}\,\{L_{aa}\mathrm{Id}\,\}$.
3. $\mathcal{E}^{bd}_{2,m} = \mathrm{Span}\,\{L_{aa}L_{bb}\mathrm{Id}\,,\ L_{ab}L_{ab}\mathrm{Id}\,,\ R_{abba}\mathrm{Id}\,,\ R_{amma}\mathrm{Id}\,,\ E\}$.

Lemma 3.1.11 suffices to describe the boundary integrands in the invariants a_1, a_2, and a_3 for Dirichlet boundary conditions. Additional structures appear for the other boundary conditions, but the analysis is similar.

3.1.9 Dimension Shifting

As with the heat content asymptotics, when the heat trace asymptotics are expressed relative to a Weyl basis, the coefficients are universal, apart from a multiplicative normalizing constant. We illustrate this principle for the interior invariants.

Lemma 3.1.12 *There exist universal constants c_i which are independent of the dimension of the underlying manifold and of the rank of the vector bundle in question so that if D is admissible on a closed Riemannian manifold M, then*

1. $a_0(F,D) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\,(F)dx$.
2. $a_2(F,D) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\,\{F(c_1E + c_2\tau\mathrm{Id}\,)\}dx$.
3. $a_4(F,D) = (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\,\{F(c_3E_{;kk} + c_4\tau E + c_5E^2 + c_6\tau_{;kk}\mathrm{Id}$
$\quad +c_7\tau^2\mathrm{Id}\, + c_8|\rho|^2\mathrm{Id}\, + c_9|R|^2\mathrm{Id}\, + c_{10}\Omega_{ij}\Omega_{ij})\}dx$.

Remark 3.1.13 The presence of the normalizing constants of $\frac{1}{6}$ and $\frac{1}{360}$ is to simplify subsequent computations and is, of course, inessential. The presence of the localizing endomorphism F enables us to recover the divergence terms $E_{;kk}$ and $\tau_{;kk}$ which otherwise would be lost.

Proof: Let $r := \mathrm{Rank}\,(V)$ and $m := \dim(M)$. By Lemma 3.1.10 (1), there exists a universal constant $c_{0,m,r}$ so that

$$a_0(F,D) = (4\pi)^{-m/2}c_{0,m,r}\int_M \mathrm{Tr}\,\{F\}dx\,. \tag{3.1.a}$$

By Lemma 3.1.5, the universal constant does not depend on r so $c_{0,m,r} = c_{0,m}$. We may therefore take

$$\Delta_m := -(\partial^2_{\theta_1} + ...\partial^2_{\theta_m})$$

on the trivial line bundle over the torus $\mathbb{T}^m$ in order to determine $c_{0,m}$. We take $F = \mathrm{Id}$; as $r = 1$, $\mathrm{Tr}\,(F) = 1$. We use Lemma 3.1.3 and Equation (3.1.a) to compute

$$a_0(1,-\partial_\theta^2)=\sqrt{\pi}=(4\pi)^{-1/2}c_{0,1}\mathrm{vol}\,(S^1)=\sqrt{\pi}c_{0,1}\,.$$

This implies that $c_{0,1}=1$. More generally, we use Lemma 3.1.6 to see

$$a_0(1,\Delta_m)=a_0(1,\Delta_1)^m$$

and consequently $c_{0,m}=1$ for all m by Lemma 3.1.7. This establishes Assertion (1).

We use Lemma 3.1.10 to establish the existence of formulae of the type given in Assertions (2) and (3). The universal constants c_i are then independent of the rank of the bundle by Lemma 3.1.5 so only the dependence upon the dimension m of the underlying manifold is in question. Again, we apply the method of universal examples. Let M_1 be a closed Riemannian manifold of dimension $m-1$ and let D_1 be an operator of Laplace type on M_1. Let

$$M:=M_1\times S^1\quad\text{and}\quad D:=D_1-\partial_\theta^2\,.$$

Let $F=F(x_1)$. As structures are flat in the S^1 direction, $E=E_D=E_{D_1}$ and

$$R^M_{ijkl}=\begin{cases}R^{M_1}_{ijkl} & \text{if } i,j,k,l\le m-1,\\ 0 & \text{if any index is } m\,.\end{cases}$$

The crucial point is that invariants formed by contractions of indices are restricted from $M_1\times S^1$ to M_1 by restricting the range of summation but have the same appearance. We refer to the discussion in Section 1.7.4 for further details concerning this point. Thus, for example,

$$\tau^M=\sum_{i,j=1}^{m}R^M_{ijji}=\sum_{i,j=1}^{m-1}R^{M_1}_{ijji}=\tau^{M_1}\,.$$

Since $(4\pi)^{-1/2}\mathrm{vol}=\sqrt{\pi}$, we have

$$\begin{aligned}&a_2(F,D)\\ =\;&(4\pi)^{-m/2}\tfrac{1}{6}\int_{M_1\times S^1}\mathrm{Tr}\left\{c_{1,m}FE+c_{2,m}F\tau^M\right\}dx_1\,d\theta\\ =\;&\sqrt{\pi}(4\pi)^{-(m-1)/2}\tfrac{1}{6}\int_{M_1}\mathrm{Tr}\left\{c_{1,m}FE+c_{2,m}F\tau^{M_1}\right\}dx_1\,.\end{aligned}\tag{3.1.b}$$

We showed $a_p(1,D_2)=0$ for $p>0$. Thus by Lemma 3.1.3 and Lemma 3.1.6,

$$\begin{aligned}&a_2(F,D)=a_2(F,D_1)a_0(1,D_2)=a_2(F,D_1)\cdot\sqrt{\pi}\\ =\;&\sqrt{\pi}(4\pi)^{-(m-1)/2}\tfrac{1}{6}\int_{M_1}\mathrm{Tr}\left\{c_{1,m-1}FE+c_{2,m-1}F\tau^{M_1}\right\}dx_1\,.\end{aligned}\tag{3.1.c}$$

We use Equations (3.1.b) and (3.1.c) to prove Assertion (2) by showing

$$c_{1,m}=c_{1,m-1}\quad\text{and}\quad c_{2,m}=c_{2,m-1}\,.$$

The proof of Assertion (3) is similar and is therefore omitted. □

We remark that if $m = 1$, then $\tau = 0$ and thus $c_{2,1}$ is not uniquely specified. We therefore set $c_{2,1} := c_{2,m}$ for any $m \geq 2$. A similar remark holds for the other constants; they are only uniquely specified for large values of m; there are relations given by Theorem 1.7.4 for small values of m.

3.1.10 Variational formulae

The Lemmas in this section do not have analogues for the heat content asymptotics. We follow the discussion in [84] and proceed formally; the necessary analytic steps can be justified using the techniques of Gilkey-Smith [208]. The boundary condition is always to be held fixed.

We begin by studying conformal variations of the operator. The following property was first observed by Branson and Ørsted [95, 96] and permits one to recover the divergence terms.

Lemma 3.1.14 *Let $(D, \mathcal{B})$ be admissible. Let $D_\varepsilon := e^{-2\varepsilon f} D$ for $f \in C^\infty(M)$. Then $\partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) = (m - n) a_n(f, D_\varepsilon, \mathcal{B})$.*

Proof: We compute that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) t^{(n-m)/2} \sim \partial_\varepsilon \mathrm{Tr}_{L^2}\left\{ e^{-tD_{\varepsilon,\mathcal{B}}} \right\} \\
= \; & -t\mathrm{Tr}_{L^2}\left\{ (\partial_\varepsilon D_\varepsilon) e^{-tD_{\varepsilon,\mathcal{B}}} \right\} = 2t\mathrm{Tr}_{L^2}\left\{ f D_\varepsilon e^{-tD_{\varepsilon,\mathcal{B}}} \right\} \\
= \; & -2t\partial_t \mathrm{Tr}_{L^2}\left\{ f e^{-tD_{\varepsilon,\mathcal{B}}} \right\} \sim -2t\partial_t \sum_{n=0}^{\infty} a_n(f, D_\varepsilon, \mathcal{B}) t^{(n-m)/2} \\
\sim \; & \sum_{n=0}^{\infty} (m-n) a_n(f, D_\varepsilon, \mathcal{B}) t^{(n-m)/2}\,.
\end{aligned}$$

The Lemma follows by equating powers of t. □

We vary the 0^{th} order term to generalize Lemma 3.1.1:

Lemma 3.1.15 *Let $F \in C^\infty(\mathrm{End}(V))$ and let $(D, \mathcal{B})$ be admissible on V.*
1. Let $D_\varepsilon := D - \varepsilon F$. Then $\partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) = a_{n-2}(F, D_\varepsilon, \mathcal{B})$.
2. Let $D_\delta := D - \delta \mathrm{Id}$. Then $\partial_\delta a_n(F, D_\delta, \mathcal{B}) = a_{n-2}(F, D_\delta, \mathcal{B})$.

Proof: We apply a similar argument to that used in the proof of Lemma 3.1.14. We compute that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) t^{(n-m)/2} \sim \partial_\varepsilon \mathrm{Tr}_{L^2}\left\{ e^{-tD_{\varepsilon,\mathcal{B}}} \right\} \\
= \; & -t\mathrm{Tr}_{L^2}\left\{ (\partial_\varepsilon D_\varepsilon) e^{-tD_{\varepsilon,\mathcal{B}}} \right\} = t\mathrm{Tr}_{L^2}\left\{ F e^{-tD_{\varepsilon,\mathcal{B}}} \right\} \\
\sim \; & t \sum_{n=0}^{\infty} a_n(F, D_\varepsilon, \mathcal{B}) t^{(n-m)/2}\,.
\end{aligned}$$

As above, we equate powers of t to complete the proof of the first assertion; the second is an immediate consequence of the first. $\square$

The final variational result combines the previous two Lemmas and was first observed in [84].

Lemma 3.1.16 *Let $(D,\mathcal{B})$ be admissible on V over M, let $f\in C^\infty(M)$, and let $F\in C^\infty(\mathrm{End}\,(V))$. Then $\partial_\varepsilon a_{m-2}(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D,\mathcal{B})=0$.*

Proof: By Lemma 3.1.14 and Lemma 3.1.15,

$$\begin{aligned}&\partial_\varepsilon a_m(1,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})=0,\quad\text{and}\\&\partial_\delta a_m(1,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})=a_{m-2}(e^{-2\varepsilon f}F,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})\,.\end{aligned}$$

Consequently

$$\begin{aligned}0&=\partial_\delta\partial_\varepsilon a_m(1,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})\\&=\partial_\varepsilon\partial_\delta a_m(1,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})\\&=\partial_\varepsilon a_{m-2}(e^{-2\varepsilon f}F,e^{-2\varepsilon f}(D-\delta F),\mathcal{B})\,.\end{aligned}$$

We evaluate at $\delta=0$ to complete the proof. $\square$

3.1.11 Recursion relations

We study the twisted Dirac operator in the 1 dimensional setting. We first study the situation on the circle. The following result is a special case of a more general relationship [179]; it plays a central role in the analysis of the leading terms in the heat asymptotics as we shall see presently.

Lemma 3.1.17 *Let $b\in C^\infty(S^1)$ be real. Define operators on $C^\infty(S^1)$ by*

$$\begin{aligned}&A:=\partial_\theta-b,\quad A^*:=-\partial_\theta-b\\&D_1:=A^*A,\quad D_2:=AA^*\,.\end{aligned}$$

Let $f\in C^\infty(S^1)$. Then

1. $2\partial_t\{K(t,\theta,\theta,D_1)-K(t,\theta,\theta,D_2)\}=\partial_\theta(\partial_\theta-2b)K(t,\theta,\theta,D_1)$.

2. $(n-1)\{a_n(f,D_1)-a_n(f,D_2)\}=a_{n-2}(\partial_\theta^2f+2b\partial_\theta f,D_1)$.

Proof: Let $\{\phi_i,\lambda_i\}$ be a discrete spectral resolution for D_1. Then the corresponding spectral resolution of D_2 on $\ker(D_2)^\perp$ is given by

$$\left\{\tfrac{A\phi_i}{\sqrt{\lambda_i}},\lambda_i\right\}_{\lambda_i\neq0}\,.$$

After differentiating with respect to t, the zero spectrum plays no role. Thus we may apply Lemma 3.1.2 to see that

$$\begin{aligned}\partial_tK(t,\theta,\theta,D_1)&=-\sum_i\lambda_ie^{-t\lambda_i}\phi_i(\theta)^2=-\sum_ie^{-t\lambda_i}D_1\phi_i(\theta)\cdot\phi_i(\theta),\\\partial_tK(t,\theta,\theta,D_2)&=-\sum_ie^{-t\lambda_i}A\phi_i(\theta)\cdot A\phi_i(\theta)\,.\end{aligned}$$

Since $D_1 = -\partial_\theta^2 + \partial_\theta b + b^2$, we prove the first assertion by computing

$$\begin{aligned}
& 2\partial_t\{K(t,\theta,\theta,D_1) - K(t,\theta,\theta,D_2)\} \\
= \quad & 2\sum_i e^{-t\lambda_i}\Big\{ - D_1\phi_i \cdot \phi_i + A\phi_i \cdot A\phi_i\Big\}(\theta) \\
= \quad & 2\sum_i e^{-t\lambda_i}\Big\{\phi_i\partial_\theta^2\phi_i - (\partial_\theta b)\phi_i^2 - b^2\phi_i^2 + (\partial_\theta\phi_i)^2 - 2b\phi_i\partial_\theta\phi_i + b^2\phi_i^2\Big\}(\theta) \\
= \quad & \partial_\theta(\partial_\theta - 2b)\sum_i e^{-t\lambda_i}\phi_i^2 = \partial_\theta(\partial_\theta - 2b)K(t,\theta,\theta,D_1).
\end{aligned}$$

We integrate by parts to see that

$$\begin{aligned}
& \sum_{n=0}^{\infty}(n-1)\Big\{a_n(f,D_1) - a_n(f,D_2)\Big\}t^{(n-3)/2} \\
\sim \quad & 2\partial_t\Big\{\mathrm{Tr}_{L^2}\{fe^{-tD_1}\} - \mathrm{Tr}_{L^2}\{fe^{-tD_2}\}\Big\} \\
= \quad & \int_M \Big\{f\partial_\theta(\partial_\theta - 2b)K(t,\theta,\theta,D_1)\Big\}d\theta \\
= \quad & \int_M \Big\{(\partial_\theta + 2b)\partial_\theta f\Big\}K(t,\theta,\theta,D_1)d\theta \\
= \quad & \mathrm{Tr}_{L^2}\Big\{((\partial_\theta + 2b)\partial_\theta f)e^{-tD_1}\Big\} \\
\sim \quad & \sum_k a_k((\partial_\theta^2 + 2b\partial_\theta)f, D_1)t^{(k-1)/2}\,.
\end{aligned}$$

We now equate powers of t in the two asymptotic expansions. $\square$

We generalize Lemma 3.1.17 to the interval by imposing suitable boundary conditions. It is closely related to Lemma 2.1.15, modulo a slight change in notation.

Lemma 3.1.18 *Let $b \in C^\infty([0,\pi])$ be real. Define operators on $C^\infty([0,\pi])$ by*

$$\begin{aligned}
A &:= \partial_x - b, \quad A^* := -\partial_x - b \\
D_1 &:= A^*A, \quad D_2 := AA^*\,.
\end{aligned}$$

Let $f \in C^\infty([0,\pi])$. Let $\mathcal{B}_1\phi := \phi|_{\partial M}$ and $\mathcal{B}_2\phi := A^*\phi|_{\partial M}$ define Dirichlet and Robin boundary conditions for D_1 and D_2, respectively. Then:

1. $2\partial_t\{K(t,x,x,D_1,\mathcal{B}_1) - K(t,x,x,D_2,\mathcal{B}_2)\} = \partial_x(\partial_x - 2b)K(t,x,x,D_1,\mathcal{B}_1)$.

2. $(n-1)\{a_n(f,D_1,\mathcal{B}_1) - a_n(f,D_2,\mathcal{B}_2)\} = a_{n-2}(\partial_x^2 f + 2b\partial_x f, D_1, \mathcal{B}_1)$.

Proof: The argument to prove Lemma 3.1.18 is essentially the same as that used to prove Lemma 3.1.17 once the boundary conditions are taken into consideration. Let $\{\phi_i, \lambda_i\}$ be a discrete spectral resolution for $D_{1,\mathcal{B}_1}$. Since the eigenfunction ϕ_i satisfies Dirichlet boundary conditions

$$A^*(A\phi_i)|_{\partial M} = \lambda_i\phi_i|_{\partial M} = 0\,.$$

This shows that $\mathcal{B}_2 A\phi_i = 0$ so

$$\left\{\frac{A\phi_i}{\sqrt{\lambda_i}}, \lambda_i\right\}_{\lambda_i \neq 0}$$

is a spectral resolution of $D_{2,\mathcal{B}_2}$ on $\ker(D_{2,\mathcal{B}_2})^\perp$. The proof of Assertion (1) now follows. To prove Assertion (2), we had to integrate by parts. Since $\phi_i|_{\partial M} = 0$, ϕ_i^2 vanishes to second order on ∂M. Thus we can safely integrate by parts without introducing additional boundary contributions and the remainder of the argument is similar. $\square$

Let Δ^p be the p form valued Laplacian discussed in Section 1.2.4. The following recursion relation is due to McKean and Singer [278].

Lemma 3.1.19 *Let M be a compact orientable 2 dimensional Riemannian manifold without boundary. Let $f \in C^\infty(M)$. Then:*

1. $\partial_t\{K(t,x,x,\Delta^0) + K(t,x,x,\Delta^2) - K(t,x,x,\Delta^1)\} = -\Delta^0 K(t,x,x,\Delta^0)$.
2. $2n\{2a_{n+2}(f,\Delta^0) - a_{n+2}(f,\Delta^1)\} = a_n(f_{;kk},\Delta^0)$.

Proof: Take a discrete spectral resolution for Δ^0 of the form

$$\left\{\phi_i, \lambda_i\right\}_{i=0}^\infty \quad \text{where} \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 ....$$

Let $\star$ be the Hodge operator discussed in Section 1.5.6. Then a discrete spectral resolution of Δ^2 is given by

$$\left\{\star\,\phi_i, \lambda_i\right\}_{i=0}^\infty .$$

Since $d + \delta$ intertwines $\Delta^0 + \Delta^2$ and Δ^1,

$$\left\{\frac{d\phi_i}{\sqrt{\lambda_i}}, \lambda_i\right\}_{i=1}^\infty \cup \left\{\frac{\delta\star\phi_i}{\sqrt{\lambda_i}}, \lambda_i\right\}_{i=1}^\infty$$

is a discrete spectral resolution for Δ^1 on $\ker(\Delta^1)^\perp$ for $i > 0$. Since the Hodge operator $\star$ is a bundle isometry and $\delta = -\star d\star$, we argue as above to see

$$\begin{aligned}
&\partial_t\Big\{K(t,x,x,\Delta^0) + K(t,x,x,\Delta^2) - K(t,x,x,\Delta^1)\Big\} \\
&= \sum_{i=1}^\infty e^{-t\lambda_i}\Big\{ -\lambda_i(\phi_i^2 + *\phi_i \cdot *\phi_i) + d\phi_i \cdot d\phi_i + \delta * \phi_i \cdot \delta * \phi_i\Big\} \\
&= \sum_{i=0}^\infty e^{-t\lambda_i}\Big\{ -2\Delta^0\phi_i \cdot \phi_i + 2d\phi_i \cdot d\phi_i\Big\} = -\Delta^0 K(t,x,x,\Delta^0)\,.
\end{aligned}$$

The first assertion of the Lemma now follows. We integrate by parts and compare powers of t in the resulting asymptotic series to establish the second assertion of the Lemma. $\square$

3.2 Functorial properties II

In this section, we discuss functorial properties of the heat trace asymptotics which primarily apply to transfer boundary conditions, to transmission boundary conditions, to spectral boundary conditions, and to time-dependent processes.

3.2.1 Transmission boundary conditions

As discussed in Section 1.6.1, we consider pairs of structures

$$\begin{aligned} &M := (M_+, M_-), \quad g := (g_+, g_-), \quad f := (f_+, f_-), \\ &V := (V_+, V_-), \quad \phi := (\phi_+, \phi_-), \quad D := (D_+, D_-)\,. \end{aligned} \tag{3.2.a}$$

Here $g_\pm$ are Riemannian metrics on compact manifolds $M_\pm$, $f_\pm$ are smooth functions over $M_\pm$, $\phi_\pm$ are smooth sections to vector bundles $V_\pm$ over $M_\pm$, and $D_\pm$ are operators of Laplace type on $V_\pm$. We assume as compatibility conditions that

$$\partial M_+ = \partial M_- = \Sigma, \quad g_+|_\Sigma = g_-|_\Sigma, \tag{3.2.b}$$

$$V_+|_\Sigma = V_-|_\Sigma, \quad f_+|_\Sigma = f_-|_\Sigma\,. \tag{3.2.c}$$

Let $\nu_\pm$ be the inward unit normals of Σ in $M_\pm$; $\nu_+ + \nu_- = 0$. If U is an impedance matching endomorphism which is defined on Σ, then the *transmission boundary operator* $\mathcal{B}_U$ is defined for $\phi = (\phi_+, \phi_-)$ by setting

$$\mathcal{B}_U\phi := \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \ \oplus \ \{\nabla_{\nu_+}\phi_+|_\Sigma + \nabla_{\nu_-}\phi_-|_\Sigma - U\phi_+|_\Sigma\}\,.$$

We may decompose the heat trace asymptotics in the form

$$a_n(f, D, \mathcal{B}_U) = a_n^M(f, D) + a_n^\Sigma(f, D, \mathcal{B}_U)$$

where the interior integrand defines a_n^M and there is an additional boundary integrand defining a_n^Σ.

Lemma 2.2.3 dealt with the heat content asymptotics if we introduced an auxiliary "artificial" singularity. There is an analogous result for the heat trace asymptotics; we omit the proof in the interest of brevity.

Lemma 3.2.1 *Let D_0 be an operator of Laplace type on a closed Riemannian manifold M_0. Let Σ be a submanifold of M_0 which separates M_0 into two components M_+ and M_-. Let $D_\pm := D_0|_{M_\pm}$ and let $U = 0$. Let $f \in C^\infty(M_0)$. Then $a_n^\Sigma(f, D, \mathcal{B}_U) = 0$.*

Lemma 2.2.1 also extends to this setting.

Lemma 3.2.2 *Let D_0 be an operator of Laplace type on a compact Riemannian manifold M_0 with boundary Σ. Let $\mathcal{B}_D$ be the Dirichlet boundary operator and let $\mathcal{B}_{R(S)}$ be the Robin boundary operator defined by S. Let $M_\pm := M_0$,*

$D_\pm := D_0$, *and* $U = -2S$. *Let* f_{even} *be the even extension of* $f_0 \in C^\infty(M_0)$ *to* M. *Then*

$$a_n^\Sigma(f_{\text{even}}, D, \mathcal{B}_U) = a_n^\Sigma(f_0, D_0, \mathcal{B}_D) + a_n^\Sigma(f_0, D_0, \mathcal{B}_{R(S)}).$$

Proof: Let $u := e^{-tD_{\mathcal{B}_U}}\phi$. As in the proof of Lemma 2.2.1, we decompose $\phi = \phi_{\text{even}} + \phi_{\text{odd}}$ and $u = u_{\text{even}} + u_{\text{odd}}$ into even and odd functions of x. Then by Equation (2.2.b),

$$\begin{aligned} u(x_+;t) &= \left\{e^{-tD_{0,\mathcal{B}_{R(S)}}}\phi_{\text{even}}\right\}(x;t) + \left\{e^{-tD_{0,\mathcal{B}_D}}\phi_{\text{odd}}\right\}(x;t), \\ u(x_-;t) &= \left\{e^{-tD_{0,\mathcal{B}_{R(S)}}}\phi_{\text{even}}\right\}(x;t) - \left\{e^{-tD_{0,\mathcal{B}_D}}\phi_{\text{odd}}\right\}(x;t). \end{aligned}$$

Let $K(t, x, \tilde{x}, D_0, \mathcal{B}_{R(S)})$ and $K(t, x, \tilde{x}, D_0, \mathcal{B}_D)$ be the associated kernels on M_0. Then

$$\begin{aligned} u(x_+;t) &= \int_{M_0} \tfrac{1}{2}\Big\{(\phi(\tilde{x}_+) + \phi(\tilde{x}_-))K(t;x,\tilde{x},D,\mathcal{B}_{R(S)}) \\ &\qquad + (\phi(\tilde{x}_+) - \phi(\tilde{x}_-))K(t,x,\tilde{x},D,\mathcal{B}_D)\Big\}dx \\ u(x_-;t) &= \int_{M_0} \tfrac{1}{2}\Big\{(\phi(\tilde{x}_+) + \phi(\tilde{x}_-))K(t;x,\tilde{x},D,\mathcal{B}_{R(S)}) \\ &\qquad + (\phi(\tilde{x}_-) - \phi(\tilde{x}_+))K(t,x,\tilde{x},D,\mathcal{B}_D)\Big\}dx. \end{aligned}$$

Consequently the kernel function is given by

$$\begin{aligned} K(t,x_+,\tilde{x}_+,D,\mathcal{B}_U) &= \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_{R(S)}) + \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_D), \\ K(t,x_+,\tilde{x}_-,D,\mathcal{B}_U) &= \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_{R(S)}) - \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_D), \\ K(t,x_-,\tilde{x}_+,D,\mathcal{B}_U) &= \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_{R(S)}) - \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_D), \\ K(t,x_-,\tilde{x}_-,D,\mathcal{B}_U) &= \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_{R(S)}) + \tfrac{1}{2}K(t,x,\tilde{x},D,\mathcal{B}_D). \end{aligned}$$

Consequently, since $f_{\text{even}}(x_+) = f_{\text{even}}(x_-)$ is an even function, one has

$$\begin{aligned} &\int_{M_+} f_{\text{even}}(x_+)\mathrm{Tr}_{V_{x_+}} K(t,x_+,x_+,D,\mathcal{B}_U)dx_+ \\ +&\int_{M_-} f_{\text{even}}(x_-)\mathrm{Tr}_{V_{x_-}} K(t,x_-,x_-,D,\mathcal{B}_U)dx_- \\ =&\int_{M_0} f_o(x)\mathrm{Tr}_{V_x}\{K(t,x,x,D,\mathcal{B}_{R(S)}) + K(t,x,x,D,\mathcal{B}_D)\}dx. \end{aligned}$$

The desired result now follows by equating coefficients of t in the resulting asymptotic expansions. □

We can discuss the de Rham complex on a singular manifold using transmission boundary conditions. We adopt the notation of Lemma 1.6.3. Let $M = M_+ \cup_\Sigma M_-$. Let e_m be the inward unit normal of $\Sigma \subset M_+$ and the outward unit normal of $\Sigma \subset M_-$. If $\{e_i\}$ is a local orthonormal frame for $TM|_\Sigma$, let

$$\mathfrak{e}_j := \mathfrak{e}(e_j), \quad \mathfrak{i}_j := \mathfrak{i}(e_j), \quad \text{and} \quad \gamma_j := \mathfrak{e}_j - \mathfrak{i}_j$$

be defined by left exterior multiplication, left interior multiplication, and left Clifford multiplication. Let

$$U = (L^+_{ab} + L^-_{ab})(\mathfrak{e}_m \mathfrak{i}_m \mathfrak{i}_a \mathfrak{e}_b + \mathfrak{i}_m \mathfrak{e}_m \mathfrak{e}_a \mathfrak{i}_b)\,.$$

Since U preserves the grading $\Lambda(M) = \oplus_p \Lambda^p(M)$, it induces transmission boundary conditions for $\Delta^p = (\Delta^p_+, \Delta^p_-)$ on $C^\infty(\Lambda^p(M))$.

Lemma 3.2.3 *Adopt the notation established above. We have*

$$\sum_p (-1)^p a_n(1, \Delta^p, \mathcal{B}_U) = \begin{cases} 0 & \textit{if} \quad n \neq m, \\ \chi(M) & \textit{if} \quad n = m\,. \end{cases}$$

Proof: By Lemma 1.6.1, $(\Delta^p, \mathcal{B}_U)$ is elliptic with respect to the cone $\mathcal{C}$. Since U is self-adjoint, the adjoint boundary condition is again $\mathcal{B}_U$. Let $\mathcal{B}_0$ be the boundary operator of Equation (1.6.c). By Lemma 1.6.3, the boundary condition $\mathcal{B}_U\phi = 0$ is equivalent to the pair of boundary conditions $\mathcal{B}_0\phi = 0$ and $\mathcal{B}_0(d+\delta)\phi = 0$. Thus $\mathcal{B}_U\phi = 0$ implies $\mathcal{B}_U(d+\delta)\phi = 0$. Let

$$\begin{aligned} E_{\text{even}}\,(\lambda) &:= \oplus_k \ker(\Delta^{2k}_{\mathcal{B}_U} - \lambda \mathrm{Id}\,), \quad \text{and} \\ E_{\text{odd}}\,(\lambda) &:= \oplus_k \ker(\Delta^{2k+1}_{\mathcal{B}_U} - \lambda \mathrm{Id}\,) \end{aligned}$$

be the associated eigenspaces of the Laplacian on the forms of even and odd degrees. We then have $d + \delta : E_{\text{even}}\,(\lambda) \to E_{\text{odd}}\,(\lambda)$ is an isomorphism for $\lambda \neq 0$. Theorem 1.3.9 now extends to this setting to show the supertrace vanishes if $n \neq m$ and that the supertrace is an integer if $n = m$. Since the index is given by a local formula, it is constant under deformations. Thus we can deform the metric to a metric which is smooth on M and so that Σ is totally geodesic. The endomorphism U vanishes for such a metric. Thus singularity Σ plays no role and can be removed by Lemma 3.2.1. The desired conclusion for $n = m$ now follows from Lemma 1.3.10. $\square$

3.2.2 Transfer boundary conditions

As with transmission boundary conditions, there is a useful relationship between transfer boundary conditions and Robin boundary conditions on the doubled manifold that we may describe as follows. We adopt the notation of Section 1.6.3. We shall work in the scalar setting for the sake of simplicity; more general functorial properties for a wider class of operators could easily be established but are not necessary for the subsequent discussion.

Let M_0 be a compact Riemannian manifold with smooth boundary Σ. Let Δ_0 be the scalar Laplacian on $C^\infty(M_0)$. Set $M_\pm := M_0$ and let Δ be the associated Laplacian on M. Let $f_\pm$ be smooth functions on M_0 with no compatibility condition imposed on Σ. Fix $0 < \theta < \frac{\pi}{2}$. Let S_{++} and S_{+-} be smooth real valued functions on Σ. Let

$$\begin{aligned}
&S_{-+} := S_{+-},\\
&S_{--} := S_{++} + (\tan\theta - \cot\theta)\, S_{+-},\\
&S_\alpha := S_{++} + \tan\theta\, S_{+-} = S_{--} + \cot\ \theta\, S_{-+},\\
&S_\beta := S_{++} - \cot\ \theta S_{+-} = S_{--} - \tan\ \theta\, S_{-+}\,.
\end{aligned}$$

Define transfer boundary conditions $\mathcal{B}_S$ on M and Robin boundary conditions $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ on M_0 by setting

$$\mathcal{B}_S\phi := \left\{ \begin{pmatrix} \nabla_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla_{\nu_-} + S_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \right\}\Bigg|_\Sigma,$$

$$\mathcal{B}_\alpha := \left(\nabla_{\nu_0} + S_\alpha\right)\phi_0\Big|_\Sigma, \quad \text{and} \quad \mathcal{B}_\beta := \left(\nabla_{\nu_0} + S_\beta\right)\phi_0\Big|_\Sigma\,.$$

Lemma 3.2.4 *Let $0 < \theta < \frac{\pi}{2}$. Adopt the notation given above. Then*

$$\begin{aligned}
a_n(f, \Delta, \mathcal{B}_S) &= a_n(\cos^2\theta\ f_+ + \sin^2\theta\ f_-, \Delta_0, \mathcal{B}_\alpha)\\
&+ a_n(\sin^2\theta\ f_+ + \cos^2\theta\ f_-, \Delta_0, \mathcal{B}_\beta)\,.
\end{aligned}$$

Proof: Let $u \in C^\infty(M_0)$. There are two different extensions of u to M we will need. First, define

$$\alpha(u)(x_+) := \cos\theta\ u(x) \quad \text{and} \quad \alpha(u)(x_-) := \sin\theta\ u(x)\,.$$

The condition $\mathcal{B}_S\alpha(u) = 0$ can be reinterpreted in terms of u as meaning

$$\begin{aligned}
&(\nabla_{\nu_0} + S_{++} + \tan\theta\ S_{+-})u|_\Sigma = 0 \quad \text{and}\\
&(\nabla_{\nu_0} + S_{--} + \cot\theta\ S_{-+})u|_\Sigma = 0, \quad \text{i.e.}\\
&(\nabla_{\nu_0} + S_\alpha)u|_\Sigma = 0 \quad \text{or equivalently} \quad \mathcal{B}_\alpha u = 0\,.
\end{aligned}$$

We also define

$$\beta(u)(x_+) := -\sin\theta\ u(x) \quad \text{and} \quad \beta(u)(x_-) := \cos\theta\ u(x)\,.$$

Analogously, $\mathcal{B}_S\beta(u) = 0$ means

$$\begin{aligned}
&(\nabla_{\nu_0} + S_{++} - \cot\theta\ S_{+-})u|_\Sigma = 0 \quad \text{and}\\
&(\nabla_{\nu_0} + S_{--} - \tan\theta\ S_{-+})u|_\Sigma = 0, \quad \text{i.e.}\\
&(\nabla_{\nu_0} + S_\beta)u|_\Sigma = 0 \quad \text{or equivalently} \quad \mathcal{B}_\beta u = 0\,.
\end{aligned}$$

It is immediate from the definition that if $u, v \in C^\infty(M_0)$, then

$$\begin{aligned}
&(\alpha(u), \beta(v))_{L^2(M)} = 0,\\
&(\alpha(u), \alpha(v))_{L^2(M)} = (u, v)_{L^2(M_0)},\\
&(\beta(u), \beta(v))_{L^2(M)} = (u, v)_{L^2(M_0)}\,.
\end{aligned} \tag{3.2.d}$$

Let $\{\lambda_i, u_i\}_{i=1}^\infty$ and $\{\mu_j, v_j\}_{j=1}^\infty$ be discrete spectral resolutions of $(\Delta_0, \mathcal{B}_\alpha)$ and $(\Delta_0, \mathcal{B}_\beta)$, respectively. As θ is fixed,

$$\Delta\alpha(u_i) = \lambda_i\alpha(u_i) \quad \text{and} \quad \Delta\beta(v_j) = \mu_j\alpha(v_j)\,.$$

The discussion above shows that

$$\mathcal{B}_S\alpha(u_i) = 0 \quad \text{and} \quad \mathcal{B}_S\beta(v_j) = 0\,.$$

As $\{u_i\}_{i=1}^\infty$ and $\{v_j\}_{j=1}^\infty$ are complete orthonormal bases for $L^2(M_0)$,

$$\left\{\alpha(u_i)\right\}_{i=1}^\infty \cup \left\{\beta(v_j)\right\}_{j=1}^\infty$$

is a complete orthonormal basis for $L^2(M)$ by Display (3.2.d). Consequently

$$\left\{\lambda_i, \alpha(u_i)\right\}_{i=1}^\infty \cup \left\{\mu_j, \beta(v_j)\right\}_{j=1}^\infty$$

is a discrete spectral resolution of Δ with transfer boundary conditions $\mathcal{B}_S$. Thus we may complete the proof by computing:

$$\begin{aligned}
&\operatorname{Tr}_{L^2}(fe^{-t\Delta_{\mathcal{B}_S}}) = \int_M \left\{ \sum_i fe^{-t\lambda_i}|\alpha(u_i)|^2 + \sum_j fe^{-t\mu_j}|\beta(v_j)|^2 \right\} \\
= \quad & \int_{M_0} \sum_i (\cos^2\theta\ f_+ + \sin^2\theta\ f_-)e^{-t\lambda_i}|u_i|^2 \\
+ \quad & \int_{M_0} \sum_j (\sin^2\theta\ f_+ + \cos^2\theta\ f_-)e^{-t\mu_j}|v_j|^2 \\
= \quad & \operatorname{Tr}_{L^2}\left\{ (\cos^2\theta\ f_+ + \sin^2\theta\ f_-)e^{-t\Delta_{0,\mathcal{B}_\alpha}} \right\} \\
+ \quad & \operatorname{Tr}_{L^2}\left\{ (\sin^2\theta\ f_+ + \cos^2\theta\ f_-)e^{-t\Delta_{0,\mathcal{B}_\beta}} \right\}. \qquad \square
\end{aligned}$$

3.2.3 Time dependent processes

Let $\mathfrak{D} = \{D_t\}_{t\ge 0}$ be a time-dependent family of operators of Laplace type as discussed previously in Section 2.9. We assume given a decomposition of the boundary $\partial M = C_N \dot\sqcup C_D$ as the disjoint union of closed sets; we permit C_N or C_D to be empty. Let $\mathfrak{B} = \{\mathcal{B}_t\}_{t\ge 0}$ be a time-dependent family of boundary operators that define Dirichlet boundary conditions on C_D and, pursuant to Equation (2.9.b), a time-dependent family of Robin boundary conditions on C_N by setting

$$\mathcal{B}_t\phi := \phi\Big|_{C_D} \oplus \left\{ \phi_{;m} + Su + \sum_{r=1}^\infty t^r(\Gamma_{a,r}u_{;a} + S_r u) \right\}\Big|_{C_N}.$$

We let u be the solution of the time-dependent heat equation

$$\begin{aligned}
&(\partial_t + D_t)u = 0, \\
&\mathcal{B}_t u = 0, \\
&u|_{t=0} = \phi\,.
\end{aligned} \tag{3.2.e}$$

There is a smooth kernel function $K = K(t, x, \bar{x}, \mathfrak{D}, \mathfrak{B})$ so that

$$u(x;t) = \int_M K(t, x, \bar{x}, \mathfrak{D}, \mathfrak{B})\phi(\bar{x})d\bar{x}\,.$$

The analogue of the heat trace function becomes in this setting

$$a(f, \mathfrak{D}, \mathfrak{B})(t) := \int_M f(x)\mathrm{Tr}_{V_x}\Big\{K(t, x, x, \mathfrak{D}, \mathfrak{B})\Big\}dx\,.$$

Exactly as in the static setting, there is a complete asymptotic expansion as $t \downarrow 0$ with locally computable coefficients a_n of the form

$$a(f, \mathfrak{D}, \mathfrak{B})(t) \sim \sum_{n=0}^{\infty} a_n(f, \mathfrak{D}, \mathfrak{B})t^{(n-m)/2}\,. \qquad (3.2.f)$$

The following Lemma generalizes Lemma 3.1.6; we omit details in the interest of brevity.

Lemma 3.2.5 *For $i = 1, 2$, let $(\mathfrak{D}_i, \mathfrak{B}_i)$ be admissible on vector bundles V_i over compact Riemannian manifolds (M_i, g_i). Let M_1 be closed so no boundary condition is needed for $\mathfrak{D}_1$. Let $(M, g) := (M_1, g_1) \times (M_2, g_2)$ be the product Riemannian manifold and let $V := \pi_1^* V_1 \otimes \pi_2^* V_2$ be the tensor product bundle over M. Define $\mathfrak{D}$ and $\mathfrak{B}$ by setting*

$$\mathfrak{D} := \mathfrak{D}_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes \mathfrak{D}_2 \quad \text{and} \quad \mathfrak{B} := \mathrm{Id}_1 \otimes \mathfrak{B}_2 \quad \text{on} \quad C^\infty(V)\,.$$

Let $f = f_1 \otimes f_2$. Then

$$a_n(F, \mathfrak{D}, \mathfrak{B}) = \textstyle\sum_{n_1+n_2=n} a_{n_1}(f_1, \mathfrak{D}_1)a_{n_2}(f_2, \mathfrak{D}_2, \mathfrak{B}_2)\,.$$

We generalize the perturbation considered in Lemma 2.2.6:

Lemma 3.2.6 *Let D be an operator of Laplace type on M. Let α and β be real parameters. Let $D_t := (1 + 2\alpha t + 3\beta t^2)D$ and let $\mathcal{B}$ be static. Then:*

1. $a_2(f, \mathfrak{D}, \mathcal{B}) = a_2(f, D, \mathcal{B}) - \frac{m}{2}\alpha a_0(f, D, \mathcal{B})$.
2. $a_3(f, \mathfrak{D}, \mathcal{B}) = a_3(f, D, \mathcal{B}) - \frac{m-1}{2}\alpha a_1(f, D, \mathcal{B})$.
3. $a_4(f, \mathfrak{D}, \mathcal{B}) = a_4(f, D, \mathcal{B}) - \frac{m-2}{2}\alpha a_2(f, D, \mathcal{B}) + (\frac{m(m+2)}{8}\alpha^2 - \frac{m}{2}\beta)a_0(f, D, \mathcal{B})$.

Proof: Let $u_0 = e^{-tD_\mathcal{B}}\phi$ and let $u(x;t) := u_0(x; t + \alpha t^2 + \beta t^3)$. Then:

$$\begin{aligned} D_t u(x;t) &= (1 + 2\alpha t + 3\beta t^2)(Du_0)(x; t + \alpha t^2 + \beta t^3) \\ \partial_t u(x;t) &= (1 + 2\alpha t + 3\beta t^2)(\partial_t u_0)(x; t + \alpha t^2 + \beta t^3). \end{aligned}$$

This shows that $(\partial_t + D_t)u = 0$. Since $u|_{t=0} = u_0|_{t=0} = \phi$ and $\mathcal{B}u = 0$, u satisfies the time-dependent heat equation. Consequently we may conclude

$$K(t, x, \bar{x}, \mathfrak{D}, \mathcal{B}) = K(t + \alpha t^2 + \beta t^3, x, \bar{x}, D, \mathcal{B})\,.$$

The Lemma will then follow from the expansions

$$a(f, \mathfrak{D}, \mathcal{B})(t) \sim \sum_{n=0}^{\infty} t^{(n-m)/2}(1 + \alpha t + \beta t^2)^{(n-m)/2} a_n(f, D, \mathcal{B}),$$

$$(1+\alpha t+\beta t^2)^j \sim 1+\alpha jt+(\tfrac{j(j-1)}{2}\alpha^2+j\beta)t^2+O(t^3)\,. \quad \square$$

Next we generalize Lemma 2.2.7 by making a time-dependent gauge transformation. Let D be a scalar operator of Laplace type and let $\Psi \in C^\infty(M)$. Define $\mathfrak{D}_\varrho$ and $\mathfrak{B}_\varrho$ by setting

$$\begin{aligned} &D_{t,\varrho} := e^{-t\varrho\Psi} D e^{t\varrho\Psi} + \varrho\Psi, \\ &\mathcal{B}_{t,\varrho}\phi := \phi|_{C_D} \oplus \{\phi_{;m} + S\phi + t\varrho\Psi_{;m}\phi\}|_{C_N}\,. \end{aligned}$$

Lemma 3.2.7 *Adopt the notation established above. Then*

$$\partial_\varrho a_n(f,\mathfrak{D}_\varrho,\mathfrak{B}_\varrho)|_{\varrho=0} = -a_{n-2}(f\Psi,D,\mathcal{B}_0)\,.$$

Proof: Let $u_0 := e^{-tD_{\mathcal{B}}}\phi$ and let $u_\varrho := e^{-t\varrho\Psi}u_0$. We wish to show that u_ϱ solves the time-dependent heat equation. We verify that the evolution equation is satisfied by checking

$$\begin{aligned} (\partial_t + D_{t,\varrho})u_\varrho &= e^{-t\rho\Psi}(\partial_t - \varrho\Psi)u_0 + e^{-t\varrho\Psi}(D+\varrho\Psi)u_0 \\ &= e^{-t\varrho\Psi}(\partial_t + D)u_0 = 0\,. \end{aligned}$$

Since $u_\varrho|_{t=0} = u_0|_{t=0} = \phi(x)$, the initial condition is satisfied. Clearly u_ϱ satisfies Dirichlet boundary conditions on C_D. On C_N, we have that

$$\begin{aligned} &\{u_{\varrho;m} + Su_\varrho + t\varrho\Psi_{;m}u_\varrho\}|_{C_N} \\ = \;&\{e^{-t\varrho\Psi}(u_{0;m} - t\varrho\Psi_{;m}u_0 + Su_0 + t\varrho\Psi_{;m}u_0)\}|_{C_N} = 0\,. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} &K(t,x,\tilde{x},\mathfrak{D}_\varrho,\mathfrak{B}_\varrho) = e^{-t\varrho\Psi}K(t,x,\tilde{x},D,\mathcal{B}_0), \\ &a(f,\mathfrak{D}_\varrho,\mathfrak{B}_\varrho)(t) = a(e^{-t\varrho\Psi}f,D,\mathcal{B}_0)(t) \\ &\qquad = a(f,D,\mathcal{B}_0)(t) - t\varrho a(f\Psi,D,\mathcal{B}_0)(t) + O(\varrho^2), \\ &\partial_\varrho a(f,\mathfrak{D}_\varrho,\mathfrak{B}_\varrho(t))|_{\varrho=0} = -ta(f\Psi,D,\mathcal{B}_0)(t)\,. \end{aligned}$$

We equate powers of t in the asymptotic expansions to complete the proof of the Lemma. $\square$

As in Lemma 2.2.9, we can make a coordinate transformation that mixes up the spatial and the temporal coordinates. Give $M := S^1 \times [0,1]$ a Riemannian metric ds^2_M and let $dx = gdx_1dx_2$ be the Riemannian element of volume. Let $\Xi \in C^\infty(M)$ have compact support near some point $P \in M$. Define a diffeomorphism of a neighborhood of $M \times \{0\}$ in $M \times [0,\infty)$ by setting

$$\Phi_\varrho(x_1,x_2;t) := (x_1 + t\varrho\Xi, x_2; t)\,.$$

Let Δ be the scalar Laplacian. Let $\mathfrak{D}_\varrho$ and $\mathfrak{B}_\varrho$ be defined by pulling back the parabolic operator $\partial_t+\Delta$ with boundary condition $\mathcal{B}$ using the diffeomorphism Φ_ϱ, i.e.

$$D_{t,\varrho} := \Phi_\varrho^*(\partial_t + \Delta) - \partial_t \quad\text{and}\quad \mathcal{B}_{t,\varrho} := \Phi_\varrho^*(\mathcal{B})\,.$$

Lemma 3.2.8 *Let Δ be the scalar Laplacian on $M := S^1 \times [0,1]$. Let $\mathcal{B}$ be a* *boundary condition so that $\Delta_{\mathcal{B}}$ is self-adjoint. Adopt the notation established above. Then*

$$\partial_\varrho a_n(f,\mathfrak{D}_\varrho,\mathfrak{B})|_{\varrho=0} = -\tfrac{1}{2}a_{n-2}(g^{-1}\partial_1(gf\Xi),\Delta,\mathcal{B})\,.$$

Proof: Let $u_0 := e^{-t\Delta_{\mathcal{B}}}\phi$ and let $u := \Phi_\varrho^* u_0$. By naturality, u solves the heat equation. As the operator determined by $\mathfrak{D}_\varrho$ at $t=0$ is $\Delta+$ lower order terms, the volume element is independent of the parameter ϱ. Thus

$$K(t,x_1,x_2,\bar{x}_1,\bar{x}_2,\mathfrak{D}_\varrho,\mathfrak{B}_\varrho) = K(t,x_1+\varrho t\Xi(x_1,x_2),x_2,\bar{x}_1,\bar{x}_2,\Delta,\mathcal{B})\,.$$

We set $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$ to evaluate on the diagonal and expand in a Taylor series

$$\begin{aligned}
&a(f,\mathfrak{D}_\varrho,\mathcal{B}_\varrho)(t)\\
&= \int_M f(x_1,x_2)K(t,x_1+\varrho t\Xi,x_2,x_1,x_2,\Delta,\mathcal{B})g\,dx_1dx_2\\
&= \int_M f(x_1,x_2)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})g\,dx_1dx_2\\
&+t\varrho\int_M f\Xi\partial_1^x K(t,x_1,x_2,y_1,x_2,\Delta,\mathcal{B})|_{x_1=y_1}g\,dx_1dx_2 + O(\varrho^2)\,.
\end{aligned}$$

As $\Delta_{\mathcal{B}}$ is self-adjoint and real, we may apply Lemma 3.1.2 to see that the heat kernel is symmetric, i.e.

$$K(t,x_1,x_2,\bar{x}_1,\bar{x}_2,\Delta,\mathcal{B}) = K(t,\bar{x}_1,\bar{x}_2,x_1,x_2,\Delta,\mathcal{B})\,.$$

Consequently, the linear term in the Taylor series expansion in ϱ is given by

$$\tfrac{1}{2}t\varrho\int_M f\Xi\partial_1(K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B}))g\,dx_1dx_2\,.$$

We integrate by parts to express this in the form

$$\begin{aligned}
&-\tfrac{1}{2}t\varrho\int_M g^{-1}\partial_1(gf\Xi)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})g\,dx_1dx_2\\
=\;&-\tfrac{1}{2}t\varrho a(g^{-1}\partial_1(gf\Xi),\Delta,\mathcal{B})(t)\,.
\end{aligned}$$

We equate terms in the asymptotic expansion in t. $\square$

We conclude our study of the functorial properties of time-dependent problems by deriving a property that involves commuting operators. For simplicity, we suppose ∂M empty; if ∂M is non-empty, it is necessary to assume Q preserves the eigenspaces of $D_{\mathcal{B}}$.

Lemma 3.2.9 *Let D be a self-adjoint static operator of Laplace type on a closed manifold M. Let Q be an auxiliary self-adjoint static partial differential operator of order at most 2 which commutes with D. Let $D_{t,\varrho} := D + 2t\varrho Q$. Assume $\mathfrak{D}_\varrho$ is of Laplace type for all ϱ. Then*

$$\partial_\varrho a_n(f,\mathfrak{D}_\varrho)|_{\varrho=0} = \partial_\varrho a_{n-2}(f,D+\varrho Q)|_{\varrho=0}\,.$$

Remark 3.2.10 If we take $Q = D$, then $D(\varrho) = (1+2t\varrho)D$. The same argument used to prove Lemma 3.2.6 shows that

$$\partial_\varrho a_n(f,(1+2t\varrho)D)|_{\varrho=0} = \tfrac{n-m-2}{2} a_{n-2}(f,D)\,.$$

On the other hand, rescaling shows that

$$a_{n-2}(f,(1+\varrho)D) = (1+\varrho)^{(n-m-2)/2} a_{n-2}(f,D)\,.$$

Thus we may establish Lemma 3.2.9 in this special case by computing:

$$\partial_\varrho a_{n-2}(f,(1+\varrho)D)|_{\varrho=0} \quad = \quad \tfrac{n-m-2}{2} a_{n-2}(f,D)\,.$$

Proof: Let $\mathcal{K}_1(t) := (1-t^2\varrho Q)e^{-tD}$; $\mathcal{K}_1(0)$ is the identity operator and

$$\begin{aligned}
&(\partial_t + D + 2t\varrho Q)(1-t^2\varrho Q)e^{-tD}\\
= \;& \{-2t\varrho Q - (1-t^2\varrho Q)D + D(1-t^2\varrho Q) + 2t\varrho Q(1-t^2\varrho Q)\}e^{-tD}\\
= \;& -2t^3\varrho^2 Q^2 e^{-tD}\,.
\end{aligned}$$

Because we are interested in the linear terms in ϱ, we may replace the fundamental solution of the heat equation for $D + 2t\varrho Q$ by the approximation

$$(1-\varrho t^2 Q)e^{-tD}\,.$$

A suitable generalization of Theorem 1.4.6 shows that there is an asymptotic expansion

$$\mathrm{Tr}_{L^2}(fQe^{-tD}) \sim \sum_{n=0}^{\infty} t^{(n-m-2)/2} a_n(f,Q,D) \quad \text{as} \quad t \downarrow 0\,.$$

Consequently

$$\begin{aligned}
& a(f,\mathfrak{D}_\rho)(t) = \mathrm{Tr}_{L^2}\{f(1-t^2\rho Q)e^{-tD}\} + O(\rho^2)\\
= \;& a(f,D)(t) - t^2\rho\,\mathrm{Tr}_{L^2}\{fQe^{-tD}\} + O(\rho^2)\,.
\end{aligned}$$

We equate coefficients of $t^{(n-m)/2}$ in the asymptotic expansions to see

$$\partial_\varrho a_n(f,\mathfrak{D}_\varrho)|_{\varrho=0} = -a_{n-2}(f,Q,D)\,. \tag{3.2.g}$$

Since Q and D commute,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \partial_\varrho a_n(f,D+\varrho Q)|_{\varrho=0} t^{(n-m)/2} \sim \partial_\varrho \mathrm{Tr}_{L^2}\left\{ fe^{-t(D+\varrho Q)} \right\}\Big|_{\varrho=0}\\
= \;& \mathrm{Tr}_{L^2}\left\{ -tfQe^{-tD} \right\} \sim -\sum_{n=0}^{\infty} a_n(f,Q,D)t^{(n-m)/2}\,.
\end{aligned}$$

Consequently

$$\partial_\varrho a_n(f,D+\varrho Q)|_{\varrho=0} = -a_n(f,Q,D)\,. \tag{3.2.h}$$

The Lemma now follows from Equations (3.2.g) and (3.2.h). $\square$

3.2.4 The invariants a_n^η

Let P be an operator of Dirac type on M; if the boundary of M is non-empty, we impose suitable boundary conditions for P and the associated boundary conditions for $D = P^2$. We expand:

$$\begin{aligned}
\mathrm{Tr}_{L^2}\left\{FPe^{-tP_{\mathcal{B}}^2}\right\} &\sim \sum_{n=0}^{\infty} a_n^\eta(F,P,\mathcal{B})t^{(n-m-1)/2},\\
\mathrm{Tr}_{L^2}\left\{Fe^{-tP_{\mathcal{B}}^2}\right\} &\sim \sum_{n=0}^{\infty} a_n(F,P^2,\mathcal{B})t^{(n-m)/2}.
\end{aligned}$$

These invariants are closely related.

Lemma 3.2.11 *Let $P(\varepsilon) := P - \varepsilon f\mathrm{Id}$ be a smooth 1 parameter family of operators of Dirac type on a Riemannian manifold of dimension m. If the boundary of M is non-empty, impose suitable boundary conditions. Then*

$$\begin{aligned}
\partial_\varepsilon a_n^\eta(1,P_\varepsilon,\mathcal{B}))|_{\varepsilon=0} &= (m-n)a_{n-1}(f,P^2,\mathcal{B}),\\
\partial_\varepsilon a_n(1,P_\varepsilon^2,\mathcal{B})|_{\varepsilon=0} &= 2a_{n-1}^\eta(f,P,\mathcal{B}).
\end{aligned}$$

Proof: We compute

$$\begin{aligned}
&\sum_n t^{(n-m-1)/2}\partial_\varepsilon a_n^\eta(1,P_\varepsilon,\mathcal{B})|_{\varepsilon=0} \sim \partial_\varepsilon \mathrm{Tr}_{L^2}\left\{P_\varepsilon e^{-tP_{\varepsilon,\mathcal{B}}^2}\right\}\Big|_{\varepsilon=0}\\
= \;&\mathrm{Tr}_{L^2}\left\{f(-1+2tP^2)e^{-tP_{\mathcal{B}}^2}\right\} = (-1-2t\partial_t)\mathrm{Tr}\left\{fe^{-tP_{\mathcal{B}}^2}\right\}\\
\sim \;&\sum_k (m-k-1)t^{(k-m)/2}a_k(f,P^2,\mathcal{B}).
\end{aligned}$$

We set $k = n-1$ to equate coefficients in the asymptotic expansions; Assertion (1) now follows. To prove the second assertion, we compute similarly

$$\begin{aligned}
&\sum_n t^{(n-m)/2}\partial_\varepsilon a_n(1,P_\varepsilon^2,\mathcal{B})|_{\varepsilon=0} = \partial_\varepsilon \mathrm{Tr}\left\{e^{-tP_{\varepsilon,\mathcal{B}}^2}\right\}\Big|_{\varepsilon=0}\\
= \;&\mathrm{Tr}\left\{2tfPe^{-tP_{\mathcal{B}}^2}\right\} \sim \sum_k t^{(k-m+1)/2}2a_k^\eta(f,P,\mathcal{B}).
\end{aligned}$$

We set $k = n-1$ and equate powers of t to establish Assertion (2). □

3.2.5 Spectral boundary conditions

Let γ give a vector bundle V over M a Clifford module structure. Let ∇ be a compatible connection on V. Let

$$P := \gamma_i\nabla_{e_i} + \psi_P \quad \text{and} \quad A := -\gamma_m\gamma_a\nabla_{e_a} + \psi_A \tag{3.2.i}$$

be operators of Dirac type on V and on $V|_{\partial M}$, respectively. Let $D := P^2$ and let A define spectral boundary conditions $\mathcal{B}_s$ for D; we shall present a slightly more general setting in Section 3.14, but postpone the discussion until that time to avoid unduly complicating the present discussion.

Lemma 3.2.12

1. *Let $A(\varepsilon) := A + \varepsilon\mathrm{Id}$ define $\mathcal{B}_s(\varepsilon)$. Then $a_n(F, D, \mathcal{B}_s(\varepsilon))$ is independent of the parameter ε.*
2. *The invariant $a_n(F, D, \mathcal{B}_s)$ is independent of the compatible connection chosen.*

Proof: For generic values of the parameter ε, the kernel of $A(\varepsilon)$ is trivial. For such a value, the boundary condition determined by the $\mathcal{B}_s(\varepsilon)$ is locally constant and thus independent of ε. Consequently $a_n(\varepsilon)$ is locally constant generically. Since a_n is given by a local formula, it is a smooth function of ε and hence independent of ε. This proves the first assertion; the second is immediate. □

Grubb and Seeley [235] gave a complete description of pole structure for the zeta function trace in the cylindrical setting which leads to formulae for the heat trace invariants in terms of data on the double and on the boundary.

Identify a neighborhood of ∂M in M with the collar $\mathcal{C} = \partial M \times [0, \epsilon)$. Assume the metric is product on the collar so

$$ds^2 = g_{\alpha\beta}(y)dy^\alpha \circ dy^\beta + dx^m \circ dx^m \,.$$

Let V have a Hermitian inner product so that γ is skew-symmetric and so that the compatible connection ∇ on V is Hermitian. Assume that $\nabla_{e_m}\psi_P = 0$ and that ψ_P anti-commutes with γ_m. Set

$$P := \gamma_i\nabla_{e_i} + \psi_P \quad \text{and} \quad A := -\gamma_m(\gamma_a\nabla_{e_a} + \psi_P)\,.$$

We may then apply Lemma 1.6.7 to see A is self-adjoint on $L^2(V|_{\partial M})$. We assume $\ker(A) = \{0\}$. We then have $P_{\mathcal{B}}$ is self-adjoint on $L^2(V)$.

We double the manifold M along the boundary ∂M to define a closed manifold N; since the metric is product on ∂M, the Riemannian metric extends smoothly to N. We double P^2 to define an operator D_N of Laplace type on N; since the coefficients of A are independent of the normal variable, $D_N = -\partial_m^2 + A^2$ on $\mathcal{C}$ and hence D_N is smooth on the double. We ignore the effect of the 0 spectrum which adds a bit of additional technical fuss. Define:

$$\eta(s, A) := \mathrm{Tr}\,_{L^2(\partial M)}(A(A^2)^{-\frac{s+1}{2}}), \qquad \zeta(s, A^2) := \mathrm{Tr}\,_{L^2(\partial M)}((A^2)^{-s})$$
$$\zeta(s, D_N) := \mathrm{Tr}\,_{L^2(N)}(D_N^{-s}), \qquad \zeta(s, D_M, \mathcal{B}) := \mathrm{Tr}\,_{L^2}(D_{M,\mathcal{B}}^{-s}).$$

Theorem 3.2.13 (Grubb & Seeley) *Adopt the notation established above.*

$$\zeta(s, D_{M,\mathcal{B}}) = \frac{R(s)}{\Gamma(s)}$$
$$+ \quad \left\{\frac{1}{2}\zeta(s, D_N) + \frac{1}{4}\left(\frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(s+1)} - 1\right)\zeta(s, A^2) - \frac{1}{4}\frac{\Gamma(s+\frac{1}{2})\eta(2s, A)}{\Gamma(\frac{1}{2})\Gamma(s+1)}\right\}$$

where the remainder R is regular away from $s = 0$.

We may then combine Lemmas 1.3.7 and 1.3.8 with Theorem 3.2.13 to establish the following result:

Theorem 3.2.14 (Grubb & Seeley) *Let P be an operator of Dirac type on a compact Riemannian manifold. Assume the structures are product near the boundary and express $P = \gamma_m(\partial_m + A)$ near the boundary. Then:*

1. *If n is even, then $a_n(1, D_M, \mathcal{B}) = \frac{1}{2}a_n(1, D_N) - \frac{1}{2(m-n)\Gamma(\frac{1}{2})}a^\eta_{n-1}(1, A)$.*

2. *If n is odd, then $a_n(1, D_M) = \frac{1}{4}\left(\frac{\Gamma(\frac{m-n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{m-n+2}{2})} - 1\right)a_{n-1}(1, A^2)$.*

Spectral boundary conditions are motivated by index theory. If

$$P : C^\infty(V_1) \to C^\infty(V_2)$$

is an elliptic complex of Dirac type, then one can impose spectral boundary conditions. We postpone the precise definitions until Section 3.14 in the interest of brevity. One then has the following result:

Lemma 3.2.15 *Adopt the notation established above. If $n < m$, then*

$$a_n(1, P^*P, \mathcal{B}_s) - a_n(1, PP^*, \mathcal{B}_s) = 0\,.$$

3.2.6 Higher order operators

Let P be elliptic with respect to the cone $\mathcal{C}$. Then P^k is elliptic with respect to the cone $\mathcal{C}$ as well. We shall assume for the sake of simplicity that M is closed; there are similar results for manifolds with boundary if we impose the associated boundary condition for P^k.

The following Lemma is an immediate generalization of Lemma 3.1.1 to this setting; we omit the proof in the interest of brevity.

Lemma 3.2.16 *Let P be a d^{th} order partial differential operator on $C^\infty(V)$ over a closed Riemannian manifold M of dimension m. Assume that P is elliptic with respect to the cone $\mathcal{C}$. Let F be an endomorphism of V. Then*

$$a_n(F, P - \varepsilon\mathrm{Id}) = \sum_{dk\le n} \frac{\varepsilon^k}{k!} a_{n-dk}(F, P)\,.$$

By an abuse of notation, we set

$$\Gamma(\tfrac{m-n}{d})^{-1}\Gamma(\tfrac{m-n}{kd}) := \lim_{s\to n}\{\Gamma(\tfrac{m-s}{d})^{-1}\Gamma(\tfrac{m-s}{kd})\}\,.$$

We can relate the heat trace asymptotics of P^k to the corresponding heat trace asymptotics of P.

Lemma 3.2.17 *Let P be a d^{th} order self-adjoint partial differential operator on $C^\infty(V)$ over a closed Riemannian manifold M of dimension m. Assume that P is elliptic with respect to the cone $\mathcal{C}$. Let F be an endomorphism of V. Then*

$$a_n(F, P^k) = \tfrac{1}{k}\Gamma(\tfrac{m-n}{d})^{-1}\Gamma(\tfrac{m-n}{dk})a_n(F, P)\,.$$

Proof: Suppose first that $P > 0$. Since $\zeta(s, F, P^k) = \zeta(ks, F, P)$, setting $t = ks$, we may apply Lemma 1.3.7 to see that

$$\begin{aligned}
a_n(F, P^k) &= \operatorname{res}_{s=\frac{m-n}{dk}} \{\Gamma(s)\zeta(s, F, P^k)\} \\
&= \operatorname{res}_{s=\frac{m-n}{dk}} \{\Gamma(s)\zeta(ks, F, P)\} \\
&= \tfrac{1}{k}\operatorname{res}_{t=\frac{m-n}{d}} \{\Gamma(\tfrac{t}{k})\zeta(t, F, P)\} \\
&= \tfrac{1}{k} \lim_{t \to \frac{m-n}{d}} \{\Gamma(\tfrac{t}{k})\Gamma(t)^{-1}\}\operatorname{res}_{t=\frac{m-n}{d}} \{\Gamma(t)\zeta(t, F, P)\} \\
&= \tfrac{1}{k}\Gamma(\tfrac{m-n}{d})^{-1}\Gamma(\tfrac{m-n}{dk}) a_n(F, P)\,.
\end{aligned}$$

The desired result now follows in the special case that $P > 0$.

More generally, let $P_\varepsilon := P + \varepsilon \mathrm{Id}$. By Theorem 1.3.4, $P_\varepsilon > 0$ for large values of the parameter ε. Thus

$$a_n(F, P_\varepsilon^k) = \tfrac{1}{k}\Gamma(\tfrac{m-n}{d})^{-1}\Gamma(\tfrac{m-n}{dk}) a_n(F, P_\varepsilon) \tag{3.2.j}$$

for $\varepsilon >> 0$. On the other hand, $a_n(F, P_\varepsilon^k)$ and $a_n(F, P_\varepsilon)$ are polynomial functions of the parameter ε. Thus Equation (3.2.j) holds for all ε. $\square$

The following variational principle will be useful in Section 3.16 in computing the heat trace asymptotics of fourth order operators.

Lemma 3.2.18 *Let Q be an operator of order $\frac{d}{2}$ which is elliptic with respect to the cone $\mathcal{K}$ on a closed Riemannian manifold M of dimension m. Let $Q_\varepsilon := Q + \varepsilon \mathrm{Id}$. Then $\frac{\partial^2}{\partial \varepsilon^2}\{a_n(F, Q_\varepsilon^2)\}|_{\varepsilon=0} = \{2 + 4\frac{m-n}{d}\} a_{n-d}(F, Q^2)$.*

Proof: We compute

$$\begin{aligned}
&\sum_{n=0}^\infty t^{(n-m)/d} \tfrac{\partial^2}{\partial \varepsilon^2}\left\{a_n(F, Q_\varepsilon^2)\right\}\Big|_{\varepsilon=0} \sim \tfrac{\partial^2}{\partial \varepsilon^2}\operatorname{Tr}_{L^2}\left\{F e^{-tQ_\varepsilon^2}\right\}\Big|_{\varepsilon=0} \\
= \quad & \operatorname{Tr}_{L^2}\left\{F(-2t + 4t^2Q^2)e^{-tQ^2}\right\} \\
= \quad & (-2t - 4t^2\partial_t)\operatorname{Tr}_{L^2}\left\{F e^{-tQ^2}\right\} \\
\sim \quad & \sum_{j=0}^\infty t^{(j-m+d)/d}\left\{-2 + 4\tfrac{m-j}{d}\right\} a_j(F, Q)\,.
\end{aligned}$$

We set $j + d = n$ and equate powers of t to complete the proof. $\square$

3.3 Heat trace asymptotics for closed manifolds

In this section, we determine the interior heat trace invariants. This result will be fundamental to our investigations in subsequent sections.

Theorem 3.3.1 *Let D be an operator of Laplace type on V over M where M is a closed Riemannian manifold. Let $F \in C^\infty(\mathrm{End}(V))$. Then:*

1. $a_0(F,D) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\,\{F\}dx$.
2. $a_2(F,D) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\,\{F(6E+\tau\mathrm{Id}\,)\}dx$.
3. $a_4(F,D) = (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\,\{F(60E_{;kk}+60\tau E+180E^2+12\tau_{;kk}\mathrm{Id}$
   $+5\tau^2\mathrm{Id}\,-2|\rho|^2\mathrm{Id}\,+2|R|^2\mathrm{Id}\,+30\Omega_{ij}\Omega_{ij})\}dx$.
4. $a_6(F,D) = \int_M \mathrm{Tr}\,\{F((\frac{18}{7!}\tau_{;iijj}+\frac{17}{7!}\tau_{;k}\tau_{;k}-\frac{2}{7!}\rho_{ij;k}\rho_{ij;k}-\frac{4}{7!}\rho_{jk;n}\rho_{jn;k}$
   $+\frac{9}{7!}R_{ijkl;n}R_{ijkl;n}+\frac{28}{7!}\tau\tau_{;nn}-\frac{8}{7!}\rho_{jk}\rho_{jk;nn}+\frac{24}{7!}\rho_{jk}\rho_{jn;kn}$
   $+\frac{12}{7!}R_{ijkl}R_{ijkl;nn}+\frac{35}{9\cdot 7!}\tau^3-\frac{14}{3\cdot 7!}\tau|\rho|^2+\frac{14}{3\cdot 7!}\tau|R|^2-\frac{208}{9\cdot 7!}\rho_{jk}\rho_{jn}\rho_{kn}$
   $-\frac{64}{3\cdot 7!}\rho_{ij}\rho_{kl}R_{ikjl}-\frac{16}{3\cdot 7!}\rho_{jk}R_{jnli}R_{knli}-\frac{44}{9\cdot 7!}R_{ijkn}R_{ijlp}R_{knlp}$
   $-\frac{80}{9\cdot 7!}R_{ijkn}R_{ilkp}R_{jlnp})\mathrm{Id}\,+\frac{1}{45}\Omega_{ij;k}\Omega_{ij;k}+\frac{1}{180}\Omega_{ij;j}\Omega_{ik;k}+\frac{1}{60}\Omega_{ij;kk}\Omega_{ij}$
   $+\frac{1}{60}\Omega_{ij}\Omega_{ij;kk}-\frac{1}{30}\Omega_{ij}\Omega_{jk}\Omega_{ki}-\frac{1}{60}R_{ijkn}\Omega_{ij}\Omega_{kn}-\frac{1}{90}\rho_{jk}\Omega_{jn}\Omega_{kn}$
   $+\frac{1}{72}\tau\Omega_{kn}\Omega_{kn}+\frac{1}{60}E_{;iijj}+\frac{1}{12}EE_{;ii}+\frac{1}{12}E_{;ii}E+\frac{1}{12}E_{;i}E_{;i}+\frac{1}{6}E^3$
   $+\frac{1}{30}E\Omega_{ij}\Omega_{ij}+\frac{1}{60}\Omega_{ij}E\Omega_j+\frac{1}{30}\Omega_{ij}\Omega_{ij}E+\frac{1}{36}\tau E_{;kk}+\frac{1}{90}\rho_{jk}E_{;jk}$
   $+\frac{1}{30}\tau_{;k}E_{;k}-\frac{1}{60}E_{;j}\Omega_{ij;i}+\frac{1}{60}\Omega_{ij;i}E_{;j}+\frac{1}{12}EE\tau+\frac{1}{30}E\tau_{;kk}+\frac{1}{72}E\tau^2$
   $-\frac{1}{180}E|\rho|^2+\frac{1}{180}E|R|^2)\}dx$.

The remainder of this section is devoted to the proof of this result. We shall establish the formulae for a_0, a_2, and a_4 and refer to [176, 347] for a discussion of a_6. We note that formulae for a_8 and a_{10} are available in this setting [5, 12, 354].

Lemmas 3.1.14 and 3.1.16 involve conformal deformations. As these properties play a central role in our investigation, we begin with the following variational formulae which follow from Equation (1.1.a), from Equation (1.1.b), and from Lemma 1.2.1. As the proof is straightforward, we omit details in the interest of brevity and refer to [84] for details.

Lemma 3.3.2 *Let M be an m dimensional Riemannian manifold.*

1. *Let $g_\varepsilon := e^{2f\varepsilon}g_0$ be a smooth 1 parameter conformal family of metrics.*
   (a) $\partial_\varepsilon|_{\varepsilon=0}dx_\varepsilon = mfdx$.
   (b) $\partial_\varepsilon|_{\varepsilon=0}\tau_\varepsilon = -2f\tau-2(m-1)f_{;ii}$.
   (c) $\partial_\varepsilon|_{\varepsilon=0}\tau_{\varepsilon;kk} = -4f\tau_{;kk}-2f_{;kk}\tau-2(m-1)f_{;iijj}+(m-6)f_{;i}\tau_{;i}$.
   (d) $\partial_\varepsilon|_{\varepsilon=0}\tau_\varepsilon^2 = -4f\tau^2-4(m-1)f_{;ii}\tau$.
   (e) $\partial_\varepsilon|_{\varepsilon=0}|\rho_\varepsilon|^2 = -4\rho^2-2f_{;ii}\tau-2(m-2)f_{;ij}\rho_{ij}$.
   (f) $\partial_\varepsilon|_{\varepsilon=0}|R_\varepsilon|^2 = -4R^2-8f_{;ij}\rho_{ij}$.
2. *Let $D_\varepsilon := e^{-2f\varepsilon}D_0$ be a smooth 1 parameter family of conformally equivalent operators of Laplace type on M. Then*

(a) $\partial_\varepsilon|_{\varepsilon=0}E_\varepsilon = -2fE + \frac{1}{2}(m-2)f_{;ii}$.
(b) $\partial_\varepsilon|_{\varepsilon=0}\tau_\varepsilon E_\varepsilon = -4f\tau E + \frac{1}{2}(m-2)f_{;ii}\tau - 2(m-1)f_{;ii}E$.
(c) $\partial_\varepsilon|_{\varepsilon=0}E^2_\varepsilon = -4fE^2 + (m-2)f_{;ii}E$.
(d) $\partial_\varepsilon|_{\varepsilon=0}\Omega^2_\varepsilon = -4f\Omega^2$.
(e) $\partial_\varepsilon|_{\varepsilon=0}E_{\varepsilon;kk} = -4fE_{;kk} - 2f_{;kk}E + \frac{1}{2}(m-2)f_{;iijj} + (m-6)f_{;k}E_{;k}$.

By Lemma 3.1.12, there exist universal constants so that

$$\begin{aligned}
a_0(F,D) &= (4\pi)^{-m/2}\int_M \operatorname{Tr}\Big\{F\Big\}dx,\\
a_2(F,D) &= (4\pi)^{-m/2}\tfrac{1}{6}\int_M \operatorname{Tr}\Big\{F(c_1E + c_2\tau\mathrm{Id}\,)\Big\}dx,\\
a_4(F,D) &= (4\pi)^{-m/2}\tfrac{1}{360}\int_M \operatorname{Tr}\Big\{F(c_3E_{;kk} + c_4\tau E + c_5E^2\\
&\quad + c_6\tau_{;kk}\mathrm{Id}\, + c_7\tau^2\mathrm{Id}\, + c_8|\rho|^2\mathrm{Id}\, + c_9|R|^2\mathrm{Id}\, + c_{10}\Omega_{ij}\Omega_{ij})\Big\}dx\,.
\end{aligned} \tag{3.3.a}$$

We prove Theorem 3.3.1 by evaluating the universal constants given above:

Lemma 3.3.3

1. $c_1 = 6$ *and* $c_5 = 180$.

2. $c_2 = 1$ *and* $c_4 = 60$.

3. $c_7 = 5$.

4. $c_3 = 60$, $c_6 = 12$, $c_8 = -2$, *and* $c_9 = 2$.

5. $c_{10} = 30$.

Proof: Let $D_\varepsilon := D - \varepsilon F$; $E_\varepsilon = \varepsilon F$. Lemma 3.1.15 and Equation (3.3.a) show

$$\begin{aligned}
\partial_\varepsilon|_{\varepsilon=0}a_2(1, D-\varepsilon F) &= \tfrac{1}{6}(4\pi)^{-m/2}\int_M \operatorname{Tr}\Big\{c_1F\Big\}dx\\
&= a_0(F,D) = (4\pi)^{-m/2}\int_M \operatorname{Tr}\Big\{F\Big\}dx,\\
\partial_\varepsilon|_{\varepsilon=0}a_4(1, D-\varepsilon F) &= \tfrac{1}{360}(4\pi)^{-m/2}\int_M \operatorname{Tr}\Big\{c_4F\tau + 2c_5FE\Big\}dx\\
&= a_2(F,D) = \tfrac{1}{6}(4\pi)^{-m/2}\int_M \operatorname{Tr}\Big\{c_1FE + c_2F\tau\Big\}dx\,.
\end{aligned}$$

These relations imply Assertion (1) as one has

$$c_1 = 6,\ c_5 = 180,\ \text{and } c_4 = 60c_2\,. \tag{3.3.b}$$

We apply Lemma 3.1.16 with $m = 4$ and $n = 2$. We use Lemma 3.3.2 to evaluate the relevant variational formulae. This leads to the identity

$$0 = \partial_\varepsilon|_{\varepsilon=0}a_2(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D) = \tfrac{1}{6}(4\pi)^{-2}\int_M \operatorname{Tr}\Big\{F(c_1 - 6c_2)f_{;ii}\Big\}dx\,.$$

This shows $c_1 = 6c_2$. Combining this identity with Equation (3.3.b) establishes Assertion (2) by showing that

$$c_2 = 1 \quad \text{and} \quad c_4 = 60\,.$$

Let M_1 and M_2 be closed Riemannian manifolds. Let $M := M_1 \times M_2$. Let Δ_M, Δ_{M_1}, and Δ_{M_2} be the associated scalar Laplacians. By Lemma 1.2.6, $E = \Omega = 0$ for the scalar Laplacian. We have

$$\begin{array}{ll} \tau_M = \tau_{M_1} + \tau_{M_2}, & \tau_{M;kk} = \tau_{M_1;kk} + \tau_{M_2;kk}, \\ |\rho_M|^2 = |\rho_{M_1}|^2 + |\rho_{M_2}|^2, & |R_M|^2 = |R_{M_1}|^2 + |R_{M_2}|^2, \\ \tau_M^2 = \tau_{M_1}^2 + \tau_{M_2}^2 + 2\tau_{M_1}\tau_{M_2}\,. & \end{array}$$

We focus on the cross term $\int_{M_1} \tau_{M_1} dx_1 \cdot \int_{M_2} \tau_{M_2} dx_2$ to express

$$a_4(1, \Delta_M) = 2c_7 \tfrac{1}{360}(4\pi)^{-m} \int_{M_1} \tau_{M_1} dx_1 \cdot \int_{M_2} \tau_{M_2} dx_2 + \dots\,.$$

Since $\Delta_M = \Delta_{M_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{M_2}$, by Lemma 3.1.6

$$\begin{aligned} a_4(1, \Delta_M) &= a_2(1, \Delta_{M_1}) a_2(1, \Delta_{M_2}) + \dots \\ &= c_2^2 \tfrac{1}{36}(4\pi)^{-m/2} \int_{M_1} \tau_{M_1} dx_1 \cdot \int_{M_2} \tau_{M_2} dx_2 + \dots\,. \end{aligned}$$

Equate the two expressions for $a_4(1, \Delta)$ to establish Assertion (3) by showing

$$c_7 = 5c_2^2 = 5\,.$$

By Lemma 3.1.16 with $m = 6$ and $n = 4$ and by Lemma 3.3.2,

$$\begin{aligned} 0 &= \partial_\varepsilon|_{\varepsilon=0} a_4(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D) \\ &= \tfrac{1}{360}(4\pi)^{-3} \int_M F \cdot \mathrm{Tr}\Big\{(-2c_3 - 10c_4 + 4c_5) f_{;kk} E \\ &\qquad + (2c_3 - 10c_6) f_{;iijj} + (-8c_8 - 8c_9) f_{;ij}\rho_{ij} \\ &\qquad + (2c_4 - 2c_6 - 20c_7 - 2c_8) f_{;ii}\tau \Big\} dx\,. \end{aligned}$$

Since the smearing functions F and f are arbitrary, this yields the relations

$$\begin{array}{ll} 0 = -2c_3 - 10c_4 + 4c_5, & 0 = 2c_3 - 10c_6, \\ 0 = 2c_4 - 2c_6 - 20c_7 - 2c_8, & 0 = -8c_8 - 8c_9\,. \end{array}$$

Since $c_4 = 60$, $c_5 = 180$, and $c_7 = 5$, we solve these equations to establish Assertion (4) by checking that

$$c_3 = 60, \quad c_6 = 12, \quad c_8 = -2, \quad \text{and} \quad c_9 = 2\,.$$

Only the universal constant c_{10} remains to be determined. We take $m = 2$. For $p = 0, 1, 2$, let E_p and Ω_p be determined by the Laplacian Δ^p on p forms.

By Lemma 1.2.6 and by Poincaré duality

$$\begin{array}{ll} E_1 e_i = -\rho_{ij} e_j, & \Omega_{1,ij} e_k = R_{ijkl} e_l, \\ E_2 = E_0 = 0, & \Omega_{2,ij} = \Omega_{0,ij} = 0\,. \end{array}$$

Consequently

$$\begin{array}{l} \mathrm{Tr}\,\{E_{1;kk}\} = -\tau_{;kk}, \\ \mathrm{Tr}\,\{E_1\tau\} = -\tau^2, \\ \mathrm{Tr}\,\{E_1^2\} = \rho_{ij}\rho_{ij} = \tfrac{1}{2}\tau^2, \\ \mathrm{Tr}\,\{\Omega_{1,ij}\Omega_{1,ij}\} = R_{ijkl}R_{ijlk} = -\tau^2\,. \end{array}$$

Let Id_p be the identity on Λ^p. As $\sum_p (-1)^p \mathrm{Tr}\,(\mathrm{Id}_p) = 0$, Lemma 1.3.10 shows

$$\begin{array}{rcl} 0 & = & a_4(1,\Delta^0) - a_4(1,\Delta^1) + a_4(1,\Delta^2) \\ & = & (4\pi)^{-1}\frac{1}{6}\displaystyle\int_M \mathrm{Tr}\left\{c_4\tau E_1 + c_5 E_1^2 + c_{10}\Omega_{1,ij}\Omega_{1,ij} - c_3\tau_{;kk}\right\}dx \\ & = & (4\pi)^{-1}\frac{1}{6}(-c_4 + \frac{1}{2}c_5 - c_{10})\int_M \tau^2 dx \end{array}$$

Thus $-c_4 + \frac{1}{2}c_5 - c_{10} = 0$ so $c_{10} = 30$. □

3.4 Heat trace asymptotics for Dirichlet boundary conditions

In this section, we shall begin our study of the heat trace asymptotics for manifolds with boundary by imposing Dirichlet boundary conditions; these boundary conditions are particularly simple as there are no additional structures which are involved.

Theorem 3.4.1 *Let D be an operator of Laplace type on V over M where M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}$ define Dirichlet boundary conditions and let $F \in C^\infty(\mathrm{End}\,(V))$. Then:*

1. $a_0(F,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\,\{F\}dx$.
2. $a_1(F,D,\mathcal{B}) = -(4\pi)^{-(m-1)/2}\frac{1}{4}\int_{\partial M} \mathrm{Tr}\,\{F\}dy$.
3. $a_2(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\,\{F(6E+\tau)\}dx$
   $+(4\pi)^{-m/2}\frac{1}{6}\int_{\partial M} \mathrm{Tr}\,\{2FL_{aa} - 3F_{;m}\}dy$.
4. $a_3(F,D,\mathcal{B}) = -\frac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M} \mathrm{Tr}\,\{96FE + F(16\tau + 8R_{amam}$
   $+7L_{aa}L_{bb} - 10L_{ab}L_{ab}) - 30F_{;m}L_{aa} + 24F_{;mm}\}dy$.
5. $a_4(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\,\{F(60E_{;kk} + 60\tau E + 180E^2 + 30\Omega^2$
   $+12\tau_{;kk} + 5\tau^2 - 2|\rho^2| + 2|R^2|)\}dx$
   $+(4\pi)^{-m/2}\frac{1}{360}\int_{\partial M} \mathrm{Tr}\,\{F(-120E_{;m} + 120EL_{aa} - 18\tau_{;m} + 20\tau L_{aa}$

$+4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac} + 24L_{aa:bb} + \frac{40}{21}L_{aa}L_{bb}L_{cc}$
$-\frac{88}{7}L_{ab}L_{ab}L_{cc} + \frac{320}{21}L_{ab}L_{bc}L_{ac}) + F_{;m}(-180E - 30\tau - \frac{180}{7}L_{aa}L_{bb}$
$+\frac{60}{7}L_{ab}L_{ab}) + 24F_{;mm}L_{aa} - 30F_{;iim}\}dy.$

Formulas for a_5 are available and will be discussed subsequently in Section 3.6. The interior integrands are given in Theorem 3.3.1 so we shall focus on the boundary integrands; for dimensional reasons, there are no boundary integrands in a_0 so Assertion (1) follows. In this Section, we will establish Assertions (2)-(4). The proof of Assertion (5) is similar for the most part so we refer to [84] for details.

By Lemma 3.1.11, there exist constants so

$$\begin{aligned}
a_1(F,D,\mathcal{B}) &= -\tfrac{1}{4}c_0(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\Big\{F\Big\}dy, \qquad (3.4.a)\\
a_2(F,D,\mathcal{B}) &= (4\pi)^{-m/2}\tfrac{1}{6}\int_M \mathrm{Tr}\Big\{F(6E+\tau)\Big\}dx\\
&+ (4\pi)^{-m/2}\tfrac{1}{6}\int_{\partial M}\mathrm{Tr}\Big\{c_1FL_{aa} + c_2F_{;m}\Big\}dy, \quad \text{and}\\
a_3(F,D,\mathcal{B}) &= -\tfrac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\Big\{F(c_4E + c_5\tau + c_6R_{amam}\\
&+ c_7L_{aa}L_{bb} + c_8L_{ab}L_{ab}) + c_9F_{;m}L_{aa} + c_{10}F_{;mm}\Big\}dy\,.
\end{aligned}$$

The same argument as that used to establish Lemma 3.1.12 shows, owing to the inclusion of suitable multiplicative normalizing factors involving powers of (4π), that these constants are independent of the rank of the vector bundle V and of the dimension of the underlying manifold M. We complete the proof of Theorem 3.4.1 by evaluating these universal constants. We begin with:

Lemma 3.4.2

1. $c_0 = 1$.

2. $c_4 = 96$ *and* $c_5 = 16$.

Proof: We use Lemma 3.1.3 to see that $a_1(1, -\partial_x^2, \mathcal{B}) = -\frac{1}{2}$ for $M := [0,\pi]$. Since ∂M consists of 2 points, the volume of ∂M is 2. Since we have included a normalizing constant of $-\frac{1}{4}$, the first assertion follows.

We use the product formulae of Lemma 3.1.6 to establish the second assertion. Let M_1 be a closed Riemannian manifold and let $M_2 = [0,1]$. Let D_1 be an operator of Laplace type on M_1 and let $D_2 := -\partial_x^2$ on M_2. We set

$$D := D_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2 \quad \text{on} \quad M := M_1 \times M_2\,.$$

Let $F = F(x_2)$. Since $a_n(1, D_1, \mathcal{B}) = 0$ for $n \geq 2$ and since $a_n(F, D_2) = 0$ for n odd, we use Lemma 3.1.6 to see

$$a_3(F,D,\mathcal{B}) = a_2(F,D_1)a_1(1,D_2,\mathcal{B})\,.$$

The invariants $\{R_{amam},\ L_{aa}L_{bb},\ L_{ab}L_{ab},\ F_{;m}L_{aa},\ F_{;mm}\}$ vanish for this example. Consequently, we may equate

$$\begin{aligned} & -\tfrac{1}{384}(4\pi)^{-(m-1)/2}2\int_{M_1}\operatorname{Tr}\Big\{F(c_4E+c_5\tau)\Big\}dx_1 \\ = & -\tfrac{1}{4}\cdot\tfrac{1}{6}(4\pi)^{-(m-1)/2}2\int_{M_1}\operatorname{Tr}\Big\{F(6E+\tau)\Big\}dx_1\,. \end{aligned}$$

This establishes the second assertion by showing that

$$c_4=\tfrac{6\cdot 384}{24}=96\quad\text{and}\quad c_5=\tfrac{384}{24}=16\,.\quad\square$$

To determine the remaining constants, we use the variational results of Section 3.1.10. We first extend Lemma 3.3.2 to this setting; again, we omit the proof in the interest of brevity.

Lemma 3.4.3 *Let M be a Riemannian manifold with smooth boundary of dimension m. Let $g_\varepsilon := e^{2\varepsilon f}g_0$ and $F_\varepsilon := e^{-2\varepsilon f}F$. Assume that $f|_{\partial M}=0$ and that $F|_{\partial M}=0$. Then on the boundary we have:*

1. $\partial_\epsilon|_{\epsilon=0}L_{aa}=-(m-1)f_{;m}$.
2. $\partial_\epsilon|_{\epsilon=0}R_{amam}=-L_{aa}f_{;m}+(m-1)f_{;mm}$.
3. $\partial_\epsilon|_{\epsilon=0}L_{aa}L_{bb}=-2(m-1)f_{;m}L_{aa}$.
4. $\partial_\epsilon|_{\epsilon=0}L_{ab}L_{ab}=-2f_{;m}L_{aa}$.
5. $\partial_\epsilon|_{\epsilon=0}F_{;m}=0$.
6. $\partial_\epsilon|_{\epsilon=0}F_{;m}L_{aa}=-(m-1)f_{;m}F_{;m}$.
7. $\partial_\epsilon|_{\epsilon=0}F_{;mm}=-5f_{;m}F_{;m}$.

We can now complete the proof of Theorem 3.4.1 by showing

Lemma 3.4.4

1. $c_1=2$, $c_2=-3$, $c_6=8$ *and* $c_{10}=24$.
2. $c_7=7$, $c_8=-10$ *and* $c_9=-30$.

Proof: Lemma 3.1.14 implies that

$$\partial_\varepsilon|_{\varepsilon=0}a_n(1,e^{-2\varepsilon f}D,\mathcal{B})+(n-m)a_n(f,D,\mathcal{B})=0\,. \tag{3.4.b}$$

Set $n=2$ in Equation (3.4.b) and use the formula given in Display (3.4.a) and the variational formulae of Lemmas 3.3.2 and 3.4.3 to see

$$\begin{aligned} 0 \;=\; & \tfrac{1}{6}(4\pi)^{-m/2}\int_M\Big\{3(m-2)-2(m-1)\Big\}f_{;ii}dx \\ + & \tfrac{1}{6}(4\pi)^{-m/2}\int_{\partial M}\Big\{-(m-1)c_1+(2-m)c_2\Big\}f_{;m}dy; \end{aligned}$$

the undifferentiated terms in f cancel out and the term $(2-m)c_2f_{;m}$ is the only boundary term introduced by $(2-m)a_2(f,D,\mathcal{B})$. After integrating by parts, this leads to the relation

$$0=-(m-1)c_1+(2-m)c_2-(m-4)\,.$$

We set $m = 1$ and $m = 2$ to determine c_1 and c_2.

Let $f|_{\partial M} = 0$. By use Lemma 1.1.4,

$$\begin{aligned} f_{;ii}|_{\partial M} &= (f_{;mm} + f_{:aa} - f_{;m}L_{aa})|_{\partial M} \\ &= (f_{;mm} - f_{;m}L_{aa})|_{\partial M}\,. \end{aligned} \tag{3.4.c}$$

We use this identity and Lemma 3.3.2 to see

$$\begin{aligned} &\partial_\varepsilon|_{\varepsilon=0}E = \tfrac{m-2}{2}f_{;mm} - \tfrac{m-2}{2}f_{;m}L_{aa} \quad \text{on} \quad \partial M, \\ &\partial_\varepsilon|_{\varepsilon=0}\tau = -2(m-1)f_{;mm} + 2(m-1)f_{;m}L_{aa} \quad \text{on} \quad \partial M\,. \end{aligned}$$

Set $n = 3$ in Equation (3.4.b) to see

$$\begin{aligned} 0 \;=\; & -\tfrac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\operatorname{Tr}\Big\{ \\ & \{\tfrac{m-2}{2}c_4 - 2(m-1)c_5 + (m-1)c_6 - c_{10}(m-3)\}f_{;mm} \\ & +(-\tfrac{m-2}{2}c_4 + 2(m-1)c_5 - c_6 - 2(m-1)c_7 - 2c_8 \\ & -(m-3)c_9\}f_{;m}L_{aa}\Big\}dy\,. \end{aligned} \tag{3.4.d}$$

Since $c_4 = 96$ and $c_5 = 16$, setting the coefficient of $f_{;mm}$ to zero shows

$$\begin{aligned} &0 = 48(m-2) - 32(m-1) + c_6(m-1) - c_{10}(m-3) \quad \text{so} \\ &c_6 = 8 \quad \text{and} \quad c_{10} = 24\,. \end{aligned}$$

Furthermore, setting the coefficient of $f_{;m}L_{aa}$ to zero in Equation (3.4.d) and using the values of c_4, c_5, c_6, and c_{10} previously obtained shows

$$0 = -48(m-2) + 32(m-1) - 8 - 2c_7(m-1) - 2c_8 - c_9(m-3)\,. \tag{3.4.e}$$

We complete the proof of Lemma 3.4.4 and thereby of Theorem 3.4.1 as well by establishing the final assertion. We suppose that both F and f vanish on ∂M. We apply Lemma 3.1.16 with $m = 5$ and $n = 3$ to compute

$$\begin{aligned} 0 &= \partial_\varepsilon a_3(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D, \mathcal{B}) \\ &= -(4\pi)^{-2}\tfrac{1}{384}\int_{\partial M}(-4c_9 - 5c_{10})F_{;m}f_{;m}dy\,. \end{aligned}$$

This leads to the relation $-4c_9 - 5c_{10} = 0$ so

$$c_9 = -\tfrac{5}{4}\cdot 24 = -30\,.$$

We then use the two relations of Equation (3.4.e) to determine c_7 and c_8. $\square$

3.5 Heat trace asymptotics for Robin boundary conditions

In this section, we continue our study of the heat trace asymptotics by imposing Robin boundary conditions which are defined by the corresponding *Robin* *boundary operator*

$$\mathcal{B}\phi := (\phi_{;m} + S\phi)|_{\partial M}\,.$$

Here $\phi_{;m}$ denotes the covariant derivative of ϕ with respect to the inward unit normal vector field using the connection defined by an operator D of Laplace type; S is an auxiliary endomorphism of $V|_{\partial M}$.

Theorem 3.5.1 *Let D be an operator of Laplace type on V over M where M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}$ define Robin boundary conditions and let $F \in C^\infty(\mathrm{End}\,(V))$. Then:*

1. $a_0(F,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\,\{F\}dx$.
2. $a_1(F,D,\mathcal{B}) = (4\pi)^{(1-m)/2}\frac{1}{4}\int_{\partial M}\mathrm{Tr}\,\{F\}dy$.
3. $a_2(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\,\{F(6E+\tau)\}dx$
   $+(4\pi)^{-m/2}\frac{1}{6}\int_{\partial M}\mathrm{Tr}\,\{F(2L_{aa}+12S)+3F_{;m}\}dy$.
4. $a_3(F,D,\mathcal{B}) = (4\pi)^{(1-m)/2}\frac{1}{384}\int_{\partial M}\mathrm{Tr}\,\{F(96E+16\tau+8R_{amam}$
   $+13L_{aa}L_{bb}+2L_{ab}L_{ab}+96SL_{aa}+192S^2 + F_{;m}(6L_{aa}+96S)$
   $+24F_{;mm}\}dy$.
5. $a_4(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\,\{F(60E_{;kk}+60\tau E+180E^2$
   $+30\Omega^2+12\tau_{;kk}+5\tau^2-2|\rho|^2+2|R|^2)\}dx$
   $+(4\pi)^{-m/2}\frac{1}{360}\int_{\partial M}\mathrm{Tr}\,\{F(240E_{;m}+42\tau_{;m}+24L_{aa:bb}+120EL_{aa}$
   $+20\tau L_{aa}+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}$
   $+\frac{40}{3}L_{aa}L_{bb}L_{cc}+8L_{ab}L_{ab}L_{cc}+\frac{32}{3}L_{ab}L_{bc}L_{ac}+360(SE+ES)$
   $+120S\tau+144SL_{aa}L_{bb}+48SL_{ab}L_{ab}+480S^2L_{aa}+480S^3$
   $+120S_{:aa})+F_{;m}(180E+30\tau+12L_{aa}L_{bb}+12L_{ab}L_{ab}$
   $+72SL_{aa}+240S^2)+F_{;mm}(24L_{aa}+120S)+30F_{;iim}\}dy$.

The remainder of this section is devoted to the proof of Assertions (1)-(4). We refer to [84] for the computation of a_4 as the computation of that term is quite similar. As with the Dirichlet operator, there is no boundary contribution to a_0 so Assertion (1) follows from Theorem 3.3.1.

The main new feature in the proof of Assertions (2-4) is the presence of structure defined by the auxiliary endomorphism S. This variable has weight 1. The arguments of Lemma 3.1.11 extend to this setting to show that there exist universal constants so that

$$\begin{aligned} a_1(F,D,\mathcal{B}) &= c_0(4\pi)^{-(m-1)/2}\tfrac{1}{4}\int_{\partial M}\mathrm{Tr}\left\{F\right\}dy, \qquad (3.5.a)\\ a_2(F,D,\mathcal{B}) &= (4\pi)^{-m/2}\tfrac{1}{6}\int_M \mathrm{Tr}\left\{F(6E+\tau)\right\}dx \end{aligned}$$

$$
\begin{aligned}
&+ \quad (4\pi)^{-m/2}\tfrac{1}{6}\int_{\partial M}\mathrm{Tr}\left\{c_1FL_{aa}+c_2F_{;m}+c_3FS\right\}dy,\\
a_3(F,D,\mathcal{B}) \quad &= \quad (4\pi)^{-(m-1)/2}\tfrac{1}{384}\int_{\partial M}\mathrm{Tr}\Big\{F(c_4E+c_5\tau+c_6R_{amam}\\
&+ \quad c_7L_{aa}L_{bb}+c_8L_{ab}L_{ab}+c_{11}SL_{aa}+c_{12}S^2)\\
&+ \quad F_{;m}(c_9L_{aa}+c_{13}S)+c_{10}F_{;mm}\Big\}dy\,.
\end{aligned}
$$

Although we have adopted a notational convention which parallels that of Display (3.4.a), we emphasize that the universal constants are different for Robin as opposed to Dirichlet boundary conditions.

The universal constants are dimension free. We generalize Lemma 3.4.2 to this setting:

Lemma 3.5.2 *Adopt the notation of Display* (3.5.a).

1. $c_0 = 1$.

2. $c_4 = 96$ *and* $c_5 = 16$.

Proof: Let $M := [0,\pi]$ be the interval. By Lemma 3.1.3, $a_1(1,-\partial_x^2,\mathcal{B}) = +\frac{1}{2}$. Since we have included a normalizing constant of $+\frac{1}{4}$ (instead of $-\frac{1}{4}$ as was done for Dirichlet boundary conditions), $c_0 = 1$.

Let M_1 be a closed Riemannian manifold and let $M_2 := [0,\pi]$. Let M be the Riemannian product of M_1 with M_2. Let

$$D = D_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2\,.$$

The product formulae of Lemma 3.1.6 show

$$a_3(F,D,\mathcal{B}) = a_2(F,D_1)a_1(1,D_2,\mathcal{B})$$

Assertion (2) follows from Assertion (1) and from Theorem 3.3.1 using the same argument as that used to establish Assertion (2) of Lemma 3.4.2. □

We relate Dirichlet and Neumann boundary conditions to prove:

Lemma 3.5.3 *Adopt the notation of Display* (3.5.a).

1. $c_2 = 3$ *and* $c_3 = 12$.

2. $c_{10} = 24$, $c_{12} = 192$, *and* $c_{13} = 96$.

Proof: Let $M := [0,\pi]$. Let b be a smooth real valued function on M. Let $b_x := \partial_x b$, $b_{xx} := \partial_x^2 b$, and so forth. Define

$$
\begin{aligned}
&A := \partial_x - b, && A^* := -\partial_x - b\\
&D_1 := A^*A = -\partial_x^2 + b_x + b^2, && D_2 := AA^* = -\partial_x^2 - b_x + b^2,\\
&\mathcal{B}_1\phi := \phi|_{\partial M}, && \mathcal{B}_2\phi := A^*\phi|_{\partial M}\,.
\end{aligned}
$$

We assume f vanishes identically near $x = \pi$ so only the component $x = 0$ where ∂_x is the inward normal is relevant. We have

$$E_1 = -b_x - b^2, \quad E_2 = b_x - b^2, \quad \text{and} \quad S = b\,.$$

We apply Lemma 3.1.18 to see

$$\begin{aligned} 0 &= (n-1)\{a_n(f,D_1,\mathcal{B}_1)-a_n(f,D_2,\mathcal{B}_2)\} \\ &- a_{n-2}(f_{xx}+2bf_x,D_1,\mathcal{B}_1)\,. \end{aligned} \tag{3.5.b}$$

We use Theorem 3.3.1 and Theorem 3.4.1. We set $n=2$ in Equation (3.5.b) to see

$$\begin{aligned} 0 &= (4\pi)^{-1/2}\tfrac{1}{6}\int_M 6f\Big\{E_1-E_2\Big\}dx \\ &+ (4\pi)^{-1/2}\tfrac{1}{6}\int_{\partial M}\Big\{-c_3fS+(-3-c_2)f_{;m}\Big\}dy \\ &- (4\pi)^{-1/2}\int_M\Big\{f_{xx}+2bf_x\Big\}dx \\ &= (4\pi)^{-1/2}\tfrac{1}{6}\int_M\Big\{-12fb_x-6f_{xx}-12bf_x\Big\}dx \\ &+ (4\pi)^{-1/2}\tfrac{1}{6}\int_{\partial M}\Big\{-c_3fb+(-3-c_2)f_{;m}\Big\}dy \\ &= (4\pi)^{-1/2}\tfrac{1}{6}\int_{\partial M}\Big\{(12-c_3)fb+(3-c_2)f_{;m}\Big\}dy\,. \end{aligned}$$

Assertion (1) now follows as we may set

$$12-c_3=0 \quad\text{and}\quad 3-c_2=0\,.$$

The proof of Assertion (2) is similar. The normalizing factors for Dirichlet and Robin boundary conditions are $-\frac{1}{384}$ and $+\frac{1}{384}$, respectively. We take $n=3$ in Equation (3.5.b) and compute

$$\begin{aligned} 0 &= -\tfrac{2}{384}\int_{\partial M}\Big\{96(E_1+E_2)+c_{12}fb^2+c_{13}f_{;m}b+(24+c_{10})f_{;mm}\Big\}dy \\ &+ \tfrac{1}{4}\int_{\partial M}\Big\{f_{;mm}+2bf_{;m}\Big\}dy \\ &= -\tfrac{1}{192}\int_{\partial M}\Big\{(c_{12}-192)fb^2+(c_{10}-24)f_{;mm}+(c_{13}-96)f_{;m}b\Big\}dy\,. \end{aligned}$$

The proof of the second assertion now follows by setting the coefficients of fb^2, $f_{;mm}$, and $f_{;m}b$ to zero. $\square$

We now consider conformal variations. Let $g_\varepsilon := e^{2\varepsilon f}g_0$ be a smooth 1 parameter family of conformally equivalent metrics. To simplify the discussion, we suppose $f|_{\partial M}=0$ so the normal vector field is unchanged on the boundary. Since the connections change, we must adjust the 0^{th} order term S to ensure the boundary conditions are held constant. This is a crucial new feature not present for Dirichlet boundary conditions.

Lemma 3.5.4 *Let M be a Riemannian manifold with smooth boundary of dimension m. Let $g_\varepsilon := e^{2\varepsilon f}g_0$, $D_\varepsilon := e^{-2\varepsilon f}D_0$, and $\mathcal{B}_0\phi := (\nabla_{e_m}+S_0)\phi|_{\partial M}$. Define S_ε by requiring that $\mathcal{B}_0\phi = (\nabla_{\varepsilon,m}+S_\varepsilon)\phi|_{\partial M}$. Assume $f|_{\partial M}=0$. Then:*

1. $\partial_\varepsilon|_{\varepsilon=0} S_\varepsilon = \frac{1}{2}(m-2)f_{;m}$.

2. $\partial_\varepsilon|_{\varepsilon=0} S_\varepsilon^2 = (m-2)f_{;m}S$.

3. $\partial_\varepsilon|_{\varepsilon=0} S_\varepsilon L_{\varepsilon,aa} = \frac{1}{2}(m-2)f_{;m}L_{aa} - (m-1)Sf_{;m}$.

Proof: We use Lemma 1.2.1 to see that on the boundary, the connection 1 form is given by:

$$\omega_{\varepsilon,m} = \omega_{0,m} + \tfrac{1}{2}(\Gamma_{\varepsilon,iim} - \Gamma_{0,iim})\mathrm{Id} = \omega_{0,m} + \tfrac{\varepsilon}{2}(2-m)f_{;m}\mathrm{Id}\ . \qquad (3.5.c)$$

To counteract the change in the connection 1 form, we therefore set

$$S_\varepsilon := S_0 + \tfrac{\varepsilon}{2}(m-2)f_{;m}\ .$$

This proves Assertions (1) and (2). We use Lemma 3.4.3 to derive Assertion (3) as $\partial_\varepsilon|_{\varepsilon=0} L_{aa} = -(m-1)f_{;m}$. $\square$

We complete the proof of Theorem 3.5.1 by using the variational formulae of Section 3.1.10 to establish the following Lemma:

Lemma 3.5.5

1. $c_9 = 6$.

2. $c_1 = 2$.

3. $c_6 = 8$ *and* $c_{11} = 96$.

4. $c_7 = 13$ *and* $c_8 = 2$.

Proof: We use Lemmas 3.3.2, 3.4.3, and 3.5.4 throughout. To prove Assertion (1), we use Lemma 3.1.16 with $m = 5$ and $n = 3$. We assume

$$f|_{\partial M} = F|_{\partial M} = 0$$

and study the coefficient of $f_{;m}F_{;m}$ to see

$$\begin{aligned} 0 &= \partial_\varepsilon a_3(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D, \mathcal{B}) \\ &= \tfrac{1}{384}(4\pi)^{-2}\int_{\partial M}(-4c_9 - 5c_{10} + \tfrac{3}{2}c_{13})f_{;m}F_{;m}dy\ . \end{aligned}$$

Consequently $0 = -4c_9 - 5c_{10} + \frac{3}{2}c_{13}$. We set $c_{10} = 24$ and $c_{13} = 96$ to establish Assertion (1) by showing

$$c_9 = 6\ .$$

We use Lemma 3.1.14 to see

$$0 = \partial_\varepsilon a_n(1, e^{-2\varepsilon f}D, \mathcal{B}) - (m-n)a_{n-2}(f, e^{-2\varepsilon f}D, \mathcal{B})\ . \qquad (3.5.d)$$

We set $n = m = 2$ and compute

$$0 = \tfrac{1}{6}(4\pi)^{-1}\int_M\Big\{-2f_{;ii}\Big\}dx + \tfrac{1}{6}(4\pi)^{-1}\int_{\partial M}\Big\{-c_1 f_{;m}\Big\}dy\ .$$

We integrate by parts and set the coefficient of $f_{;m}$ to zero to prove the second assertion by checking that

$$c_1 = 2\ .$$

Next, we set $n = 3$ in Equation (3.5.d) and use Equation (3.4.c) to see

$$\begin{aligned} 0 &= -(4\pi)^{-(m-1)/2}\tfrac{1}{384}\int_{\partial M}\Big\{(-\tfrac{m-2}{2}c_4 + 2(m-1)c_5 - c_6 \\ &\quad -2(m-1)c_7 - 2c_8 + \tfrac{m-2}{2}c_{11} - (m-3)c_9)f_{;m}L_{aa} \\ &\quad +(\tfrac{m-2}{2}c_4 - 2(m-1)c_5 + (m-1)c_6 - (m-3)c_{10})f_{;mm} \\ &\quad +(-(m-1)c_{11} + (m-2)c_{12} - (m-3)c_{13})f_{;m}S\Big\}dy\,. \qquad (3.5.e) \end{aligned}$$

Substitute the previously determined values $c_4 = 96$, $c_5 = 16$, $c_{10} = 24$, $c_{12} = 192$, and $c_{13} = 96$ into this relation. Setting the coefficients of $f_{;mm}$ and $f_{;m}S$ equal to zero then yields the equations

$$\begin{aligned} &0 = 48(m-2) - 32(m-1) + c_6(m-1) - 24(m-3) \\ &0 = -(m-1)c_{11} + 192(m-2) - 96(m-3)\,. \end{aligned}$$

This establishes Assertion (3).

We have $c_9 = 6$ and $c_{11} = 96$. We set the coefficient of $f_{;m}L_{aa}$ to zero in Equation (3.5.e) to obtain the relation

$$\begin{aligned} 0 &= -48(m-2) + 32(m-1) - 8 - 2(m-1)c_7 - 2c_8 \\ &+ 48(m-2) - 6(m-3)\,. \end{aligned}$$

Since m is arbitrary, this yields two separate equations

$$0 = 26 - 2c_7 \quad \text{and} \quad 0 = -22 + 2c_7 - 2c_8\,.$$

We solve these two equations to determine c_7 and c_8 and thereby complete the proof. □

3.6 Heat trace asymptotics for mixed boundary conditions

We can combine Theorems 3.4.1 and 3.5.1 into a single result by using the mixed boundary conditions introduced in Section 1.5.3. We assume given a decomposition

$$V|_{\partial M} = V_+ \oplus V_-\,. \qquad (3.6.a)$$

Our first task is to extend the bundles $V_\pm$ to a neighborhood of ∂M in M. Let $\Pi_\pm$ be the complementary projections on $V_\pm$ defined by the decomposition of Equation (3.6.a). We extend the endomorphisms $\Pi_\pm$ to be parallel with respect to the inward unit geodesic normal vector field on a collared neighborhood of ∂M. The relations

$$\Pi_+ + \Pi_- = \mathrm{Id}\,, \quad \Pi_\pm^2 = \Pi_\pm, \quad \text{and} \quad \Pi_\pm\Pi_\mp = 0$$

are preserved by parallel translation and thus extend to the collared neighborhood. The desired extensions of the bundles $V_\pm$ to a neighborhood of the boundary are then given by defining

$$V_\pm := \text{range}\,(\Pi_\pm)\,.$$

To encode the splitting of Equation (3.6.a) into a single endomorphism, set

$$\chi := \Pi_+ - \Pi_-\,.$$

We then have that

$$\Pi_\pm = \tfrac{1}{2}(\text{Id}\,\pm\chi) \quad \text{and} \quad V_\pm = \text{range}\,(\tfrac{1}{2}(\text{Id}\,\pm\chi))\,.$$

Let S be an auxiliary endomorphism of V_+ over ∂M. The *mixed boundary operator* may then be defined by setting

$$\mathcal{B}\phi := \Pi_+(\phi_{;m} + S\phi)|_{\partial M} \oplus \Pi_-\phi|_{\partial M}\,.$$

Theorem 3.6.1 *Let D be an operator of Laplace type on V over M where M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B} = \mathcal{B}(\chi, S)$ define mixed boundary conditions and let $F = f\cdot\text{Id}$ be a scalar operator where $f \in C^\infty(M)$. Then:*

1. $a_0(F, D, \mathcal{B}) = (4\pi)^{-m/2}\int_M \text{Tr}\,\{F\}dx.$
2. $a_1(F, D, \mathcal{B}) = (4\pi)^{-(m-1)/2}\frac{1}{4}\int_{\partial M} \text{Tr}\,\{F\chi\}dy.$
3. $a_2(F, D, \mathcal{B}) = (4\pi)^{-m/2}\frac{1}{6}\int_M \text{Tr}\,\{F(6E+\tau)\}dx$
$+(4\pi)^{-m/2}\frac{1}{6}\int_{\partial M} \text{Tr}\,\{2FL_{aa} + 3F_{;m}\chi + 12FS\}dy.$
4. $a_3(F, D, \mathcal{B}) = (4\pi)^{-(m-1)/2}\frac{1}{384}\int_{\partial M} \text{Tr}\,\{F(96\chi E + 16\chi\tau + 8\chi R_{amam}$
$+[13\Pi_+ - 7\Pi_-]L_{aa}L_{bb} + [2\Pi_+ + 10\Pi_-]L_{ab}L_{ab} + 96SL_{aa}$
$+192S^2 - 12\chi_{:a}\chi_{:a}) + F_{;m}([6\Pi_+ + 30\Pi_-]L_{aa} + 96S)$
$+24\chi F_{;mm}\}dy.$
5. $a_4(F, D, \mathcal{B}) = (4\pi)^{-m/2}\frac{1}{360}\int_M \text{Tr}\,\{F(60E_{;kk} + 60\tau E + 180E^2 + 30\Omega^2$
$+12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2)\}dx$
$+(4\pi)^{-m/2}\frac{1}{360}\int_M \text{Tr}\,\{F([240\Pi_+ - 120\Pi_-]E_{;m} + [42\Pi_+ - 18\Pi_-]\tau_{;m}$
$+120EL_{aa} + 24L_{aa:bb} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab}$
$+4R_{abcb}L_{ac} + 720ES + 120S\tau + [\frac{280}{21}\Pi_+ + \frac{40}{21}\Pi_-]L_{aa}L_{bb}L_{cc}$
$+[\frac{168}{21}\Pi_+ - \frac{264}{21}\Pi_-]L_{ab}L_{ab}L_{cc} + [\frac{224}{21}\Pi_+ + \frac{320}{21}\Pi_-]L_{ab}L_{bc}L_{ac}$
$+144SL_{aa}L_{bb} + 48SL_{ab}L_{ab} + 480S^2L_{aa} + 480S^3 + 120S_{:aa}$
$+60\chi\chi_{:a}\Omega_{am} - 12\chi_{:a}\chi_{:a}L_{bb} - 24\chi_{:a}\chi_{:b}L_{ab} - 120\chi_{:a}\chi_{:a}S)$
$+F_{;m}(180\chi E + 30\chi\tau + [\frac{84}{7}\Pi_+ - \frac{180}{7}\Pi_-]L_{aa}L_{bb} + 240S^2$
$+[\frac{84}{7}\Pi_+ + \frac{60}{7}\Pi_-]L_{ab}L_{ab} + 72SL_{aa} - 18\chi_{:a}\chi_{:a})$
$+F_{;mm}(24L_{aa} + 120S) + 30F_{;iim}\chi\}dy.$

We present the formula for a_5 in Section 3.6.2 below to avoid encumbering the discussion with a lengthy formula.

3.6.1 The proof of Theorem 3.6.1

The interior integrands in Theorem 3.6.1 are determined by Theorem 3.3.1. Thus we must determine the boundary integrands for mixed boundary conditions to establish Theorem 3.6.1. We follow the discussion in [84] after correcting an error in the calculation given there which was pointed out subsequently by Vassilevich. See the discussion in [161].

We say that D respects the splitting of $V = V_+ \oplus V_-$ near the boundary if there exist operators $D_\pm$ of Laplace type on $V_\pm$ near ∂M so that

$$D = \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix}.$$

If this happens, then the structures decouple and we may decompose

$$\mathcal{B} = \mathcal{B}_{R(S)} \oplus \mathcal{B}_D$$

where $\mathcal{B}_{R(S)}$ defines Robin boundary conditions on $V_+|_{\partial M}$ and $\mathcal{B}_D$ defines Dirichlet boundary conditions on $V_-|_{\partial M}$. In this setting, we may use Lemma 3.1.5 to see

$$a_n^{\partial M}(F, D, \mathcal{B}) = a_n^{\partial M}(F, D_+, \mathcal{B}_{R(S)}) + a_n^{\partial M}(F, D_-, \mathcal{B}_D). \tag{3.6.b}$$

We emphasize that the splitting $V = V_+ \oplus V_-$ is defined only near the boundary and thus this decomposition of D need hold only near the boundary; the boundary integrands are local.

The additional invariants involving the tangential derivatives of χ measure the failure of D to split near the boundary. They enter only at the a_3 level. Consequently, we may use Theorem 3.4.1, Theorem 3.5.1, and Equation (3.6.b) to establish Assertions (1-3) of Theorem 3.6.1.

As $\chi^2 = \mathrm{Id}$,

$$\chi\chi_{:a} + \chi_{:a}\chi = 0 \quad \text{and} \quad \chi\chi_{:aa} + 2\chi_{:a}\chi_{:a} + \chi_{:aa}\chi = 0. \tag{3.6.c}$$

Because F is scalar, F commutes with the various endomorphisms involved. We write down Weyl spanning sets to see there exist universal constants c_i so

$$\begin{aligned}
a_3(F, D, \mathcal{B}) &= (4\pi)^{-(m-1)/2}\tfrac{1}{384}\int_{\partial M} \mathrm{Tr}\Big\{F(96\chi E + 16\chi\tau \\
&+ 8\chi R_{amam} + [13\Pi_+ - 7\Pi_-]L_{aa}L_{bb} + 96SL_{aa} \\
&+ [2\Pi_+ + 10\Pi_-]L_{ab}L_{ab} + 192S^2 + c_1\chi_{:a}\chi_{:a}) \\
&+ F_{;m}([6\Pi_+ + 30\Pi_-]L_{aa} + 96S) + 24\chi F_{;mm}\Big\}dy,
\end{aligned} \tag{3.6.d}$$

$$\begin{aligned}
a_4(F, D, \mathcal{B}) &= (4\pi)^{-m/2}\tfrac{1}{360}\int_M \mathrm{Tr}\Big\{F(60E_{;kk} + 60\tau E + 180E^2 \\
&+ 30\Omega^2 + 12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2)\Big\}dx
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (4\pi)^{-m/2}\tfrac{1}{360}\int_{\partial M}\mathrm{Tr}\Big\{F([240\Pi_+ - 120\Pi_-]E_{;m}\\
&+ \quad [42\Pi_+ - 18\Pi_-]\tau_{;m} + 24L_{aa:bb} + 120EL_{aa}\\
&+ \quad 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab}\\
&+ \quad 4R_{abcb}L_{ac} + [\tfrac{280}{21}\Pi_+ + \tfrac{40}{21}\Pi_-]L_{aa}L_{bb}L_{cc}\\
&+ \quad [\tfrac{168}{21}\Pi_+ - \tfrac{264}{21}\Pi_-]L_{ab}L_{ab}L_{cc} + 720SE + 480S^3\\
&+ \quad [\tfrac{224}{21}\Pi_+ + \tfrac{320}{21}\Pi_-]L_{ab}L_{bc}L_{ac} + 120S\tau + 120S_{:aa}\\
&+ \quad 144SL_{aa}L_{bb} + 48SL_{ab}L_{ab} + 480S^2L_{aa}\\
&+ \quad c_2\chi\chi_{:a}\Omega_{am} + c_3\chi_{:a}\chi_{:a}L_{bb} + c_4\chi_{:a}\chi_{:b}L_{ab}\\
&+ \quad c_5\chi_{:a}\chi_{:a}S + c_7\chi_{:a}\Omega_{am}) + 30\chi F_{;iim}\\
&+ \quad F_{;m}(240S^2 + 30\chi\tau + [\tfrac{84}{7}\Pi_+ - \tfrac{180}{7}\Pi_-]L_{aa}L_{bb}\\
&+ \quad 72SL_{aa} + 180\chi E + [\tfrac{84}{7}\Pi_+ + \tfrac{60}{7}\Pi_-]L_{ab}L_{ab}\\
&+ \quad c_6\chi_{:a}\chi_{:a}) + F_{;mm}(24L_{aa} + 120S)\Big\}dy\,.
\end{aligned}
$$

A few additional remarks are in order concerning terms which do not appear. The fact that f is scalar is crucial as this permits to cyclically permute the trace. Since $\mathrm{Tr}\,\chi = \dim V_+ - \dim V_-$ is constant,

$$\mathrm{Tr}\,(\chi_{:aa}) = \mathrm{Tr}\,(\chi)_{:aa} = 0\,.$$

One can use Equation (3.6.c) and then cyclically permute the entries in the trace to see

$$
\begin{aligned}
&\mathrm{Tr}\,(\chi\chi_{:a}\chi_{:a}) = -\mathrm{Tr}\,(\chi_{:a}\chi\chi_{:a}) = -\mathrm{Tr}\,(\chi\chi_{:a}\chi_{:a}) \quad \text{so}\\
&\mathrm{Tr}\,(\chi\chi_{:a}\chi_{:a}) = 0\,.
\end{aligned}
$$

We also have $\mathrm{Tr}\,(\chi\chi_{:aa}) = \mathrm{Tr}\,(\chi_{:aa}\chi) = -\mathrm{Tr}\,(\chi_{:a}\chi_{:a})$. Thus we have omitted the following invariants from consideration in a_3

$$\{\mathrm{Tr}\,(f\chi_{:aa}), \mathrm{Tr}\,(f\chi\chi_{:aa}), \mathrm{Tr}\,(f\chi\chi_{:a}\chi_{:a})\}\,.$$

Similar arguments are used in the study of a_4. As $\mathrm{Tr}\,(\chi\chi_{:a}) = \mathrm{Tr}\,(\chi_{:a}) = 0$, the invariants

$$\{\mathrm{Tr}\,(f\chi_{:a}L_{ab:b}), \mathrm{Tr}\,(f\chi_{:a}L_{bb:a}), \mathrm{Tr}\,(f\chi\chi_{:a}L_{ab:b}), \mathrm{Tr}\,(f\chi\chi_{:a}L_{bb:a})\}$$

do not appear in a_4. The most difficult invariants to eliminate are those involving S. Since $\chi S = S\chi = S$, one has

$$
\begin{aligned}
&\mathrm{Tr}\,(\chi_{:aa}S) = \mathrm{Tr}\,(\chi\chi_{:aa}S) = \mathrm{Tr}\,(\chi_{:aa}\chi S) = \mathrm{Tr}\,(\chi_{:a}\chi_{:a}S),\\
&\mathrm{Tr}\,(\chi\chi_{:a}\chi_{:a}S) = \mathrm{Tr}\,(\chi_{:a}\chi_{:a}S)\,.
\end{aligned}
$$

One has

$$\mathrm{Tr}\,(\chi_{:a}S) = \mathrm{Tr}\,(\chi_{:a}\chi S) = -\mathrm{Tr}\,(\chi\chi_{:a}S) = -\mathrm{Tr}\,(\chi_{:a}S)$$

so $\mathrm{Tr}\,(\chi_{:a}S) = 0$. Covariantly differentiating this identity yields

$$\begin{aligned}&\mathrm{Tr}\,(\chi_{:aa}S) + \mathrm{Tr}\,(\chi_{:a}S_{:a}) = 0 \quad \text{so}\\ &\mathrm{Tr}\,(\chi_{:a}S_{:a}) = -\mathrm{Tr}\,(\chi_{:aa}S) = -\mathrm{Tr}\,(\chi_{:a}\chi_{:a}S)\,.\end{aligned}$$

Since $\chi S = S$, $\chi_{:a}S + \chi S_{:a} = S_{:a}$. Consequently,

$$\begin{aligned}&\mathrm{Tr}\,(\chi_{:a}\chi_{:a}S + \chi_{:a}\chi S_{:a}) = \mathrm{Tr}\,(\chi_{:a}S_{:a}) \quad \text{so}\\ &\mathrm{Tr}\,(\chi\chi_{:a}S_{:a}) = \mathrm{Tr}\,(\chi_{:a}\chi_{:a}S) - \mathrm{Tr}\,(\chi_{:a}S_{:a}) = 2\mathrm{Tr}\,(\chi_{:a}\chi_{:a}S)\,.\end{aligned}$$

Finally, since $\mathrm{Tr}\,(\chi_{:a}S + \chi S_{:a}) = \mathrm{Tr}\,(S_{:a})$ and $\mathrm{Tr}\,(\chi_{:a}S) = 0$, we have that $\mathrm{Tr}\,(\chi S_{:a}) = \mathrm{Tr}\,(S_{:a})$. Covariant differentiation yields

$$\begin{aligned}&\mathrm{Tr}\,(\chi_{:a}S_{:a}) + \mathrm{Tr}\,(\chi S_{:aa}) = \mathrm{Tr}\,(S_{:aa}) \quad \text{so}\\ &\mathrm{Tr}\,(\chi S_{:aa}) = \mathrm{Tr}\,(S_{:aa}) - \mathrm{Tr}\,(\chi_{:a}S_{:a}) = \mathrm{Tr}\,(S_{:aa}) + \mathrm{Tr}\,(\chi_{:a}\chi_{:a}S)\,.\end{aligned}$$

This shows one must also omit the following invariants from consideration when dealing with a_4

$$\begin{aligned}&\{\mathrm{Tr}\,(f\chi_{:aa}S), \mathrm{Tr}\,(f\chi\chi_{:aa}S), \mathrm{Tr}\,(f\chi\chi_{:a}\chi_{:a}S),\\ &\ \mathrm{Tr}\,(f\chi\chi_{:a}S_{:a}, \mathrm{Tr}\,(f\chi_{:a}S_{:a}), \mathrm{Tr}\,(f\chi S_{:aa})\}\,.\end{aligned}$$

We begin our computation of the unknown coefficients c_i as follows:

Lemma 3.6.2 $c_7 = 0$.

Proof: We apply Lemma 3.1.4 to study c_7. If (D, S, χ) is real, then a_4 is real. Consequently the coefficient c_7 must be real. On the other hand, if ∇ is Riemannian and if S and χ are self-adjoint, then $D_{\mathcal{B}}$ is self-adjoint. Since F is scalar, this implies a_4 is real. Since $\sqrt{-1}\Omega$ and $\chi_{:a}$ are self-adjoint, this implies c_7 is purely imaginary. $\square$

We use the Gauss-Bonnet theorem to derive additional relations.

Lemma 3.6.3

1. $c_1 = -12$.

2. $c_2 = 60$.

3. $c_5 = -120$.

Proof: Let M be a compact Riemannian manifold with smooth boundary ∂M. Let Δ^p be the Laplace-Beltrami operator on the space of smooth p forms and let $\mathcal{B}$ define absolute boundary conditions. By Lemma 1.5.10,

$$\textstyle\sum_p (-1)^p a_n(1, \Delta^p, \mathcal{B}) = 0 \quad \text{for} \quad n \neq m\,. \tag{3.6.e}$$

If $\{e_i\}$ is a local orthonormal frame, let $\mathfrak{e}_i$ and $\mathfrak{i}_i$ denote exterior multiplication by e_i and the dual, interior multiplication by e_i. Let Ω_{ij} be the curvature of the Levi-Civita connection acting on the exterior algebra. We use Lemma 1.2.5 and Lemma 1.5.4 to see

$$\begin{aligned}
&E=-\tfrac{1}{2}\gamma_i\gamma_j\Omega_{ij}, \qquad\qquad (3.6.\mathrm{f})\\
&\chi=-1 \text{ on } \Lambda(\partial M)^\perp=\operatorname{Span}\{dy^I\wedge dx^m\},\\
&\chi=+1 \text{ on } \Lambda(\partial M)=\operatorname{Span}\{dy^I\},\\
&\chi_{:a}=2L_{ab}(\mathfrak{e}_b\mathfrak{i}_m+\mathfrak{e}_m\mathfrak{i}_b),\\
&S=-\mathfrak{e}(e_a)\mathfrak{i}(e_b)L_{ab} \text{ on } \Lambda(\partial M)=\operatorname{Span}\{dy^I\}\,.
\end{aligned}$$

We let M be a compact Riemann surface. Let $\kappa:=L_{11}$ be the geodesic curvature. By Display (3.6.f),

$$\begin{aligned}
&\sum_p(-1)^p\operatorname{Tr}(L_{aa}S_p)=\kappa^2, &&\sum_p(-1)^p\operatorname{Tr}(S_p^2)=-\kappa^2,\\
&\sum_p(-1)^p\operatorname{Tr}(\chi_{:a}\chi_{:a})=-8\kappa^2, &&\sum_p(-1)^p\operatorname{Tr}(\chi_pE_p)=0,\\
&\sum_p(-1)^p\operatorname{Tr}(E_p)=\tau, &&\sum_p(-1)^p\operatorname{Tr}(\operatorname{Id}_p)=0,\\
&\sum_p(-1)^p\operatorname{Tr}(\chi_p)=0\,.
\end{aligned}$$

We may now use Equation (3.6.e) with $m=2$ and $n=3$ to see

$$\begin{aligned}
0&=\sum_p(-1)^pa_3(1,\Delta^p,\mathcal{B})\\
&=(4\pi)^{-m/2}\tfrac{1}{384}\int_{\partial M}\Big\{96-192-8c_1\Big\}\kappa^2dy\,.
\end{aligned}$$

Assertion (1) now follows as this equation implies $c_1=-12$.

Similarly by taking $m=2$ and $n=4$, we see that

$$\begin{aligned}
0=\int_{\partial M}\Big\{&(192-480+480-8c_3-8c_4+4c_5)\kappa^3\\
&+(120-360+120+2c_2)\kappa\tau\Big\}dy\,.
\end{aligned}$$

Thus $c_2=60$, which proves Assertion (3). We also obtain the relation, that we will use subsequently in the proof of Lemma 3.6.4,

$$0=48-2c_3-2c_4+c_5\,. \qquad (3.6.\mathrm{g})$$

To determine c_5, we take $m=3$, $M:=S^1\times S^1\times[0,1]$, and

$$ds_M^2=e^{2f_1}d\theta_1^2+e^{2f_2}d\theta_2^2+dr^2\,.$$

We take $f_1=f_2=0$ on ∂M. We then compute, after a bit of effort, that

$$\sum_p(-1)^p\operatorname{Tr}\{144SL_{aa}L_{bb}\}=0,\quad \sum_p(-1)^p\operatorname{Tr}\{48SL_{ab}L_{ab}\}=0,$$

$$\sum_p(-1)^p\mathrm{Tr}\,\{c_3\chi_{:a}\chi_{:a}L_{bb}\}=0,\quad \sum_p(-1)^p\mathrm{Tr}\,\{c_4\chi_{:a}\chi_{:b}L_{ab}\}=0,$$
$$\sum_p(-1)^p\mathrm{Tr}\,\{480S^2L_{aa}\}=960L_{11}L_{22}^2+\text{other terms},$$
$$\sum_p(-1)^p\mathrm{Tr}\,\{480S^3\}=-1440L_{11}L_{22}^2+\text{other terms},\tag{3.6.h}$$
$$\sum_p(-1)^p\mathrm{Tr}\,\{c_5\chi_{:a}\chi_{:a}S\}=-4c_5L_{11}L_{22}^2+\text{other terms}\,.$$

Assertion (3) now follows since $0=960-1440-4c_5$ implies $c_5=-120$. □

We remark that the four vanishing results which are given on the first two lines of Display (3.6.h) are not accidental; they follow from more general vanishing results we will establish presently in Section 3.8 in our discussion of the Witten Laplacian.

We complete the proof of Theorem 3.6.1 by showing that

Lemma 3.6.4 $c_3=-12$, $c_4=-24$, *and* $c_6=-18$.

Proof: Let $D_\varepsilon:=e^{-2f\varepsilon D}$ for $f\in C^\infty(M)$ and $f|_{\partial M}=0$. By Lemma 3.1.15,

$$\partial_\varepsilon a_4(1,D_\varepsilon,\mathcal{B})-(m-4)a_4(f,D_\varepsilon,\mathcal{B})=0\,.\tag{3.6.i}$$

Equation (3.5.c) implies that

$$\omega_{\varepsilon,j}=\omega_{0,j}+\tfrac{1}{2}(\Gamma_{\varepsilon,jji}-\Gamma_{0,jji})\mathrm{Id}\ .$$

Thus $\partial_\varepsilon\omega_\varepsilon$ is scalar so

$$\partial_\epsilon|_{\epsilon=0}\nabla\chi=[\partial_\varepsilon\omega,\chi]=0\,.$$

This makes the computation of the variations of the new invariants entirely straightforward. The only tricky point is to note that the variation of S takes the form

$$\partial_\epsilon|_{\epsilon=0}S=-fS+\tfrac{1}{2}(m-2)f_{;m}\Pi_+;$$

this was incorrectly computed in [84] and the error subsequently pointed out by Vassilevich. One has

$$\mathrm{Tr}\,(\chi_{:a}\chi_{:a}\Pi_+)=\tfrac{1}{2}\mathrm{Tr}\,(\chi_{:a}\chi_{:a}+\chi\chi_{:a}\chi_{:a})=\tfrac{1}{2}\mathrm{Tr}\,(\chi_{:a}\chi_{:a})\,.$$

Setting the coefficient of $\mathrm{Tr}\,(\chi_{:a}\chi_{:a}f_{;m})$ in Equation (3.6.i) to zero yields

$$-c_3(m-1)-c_4+\tfrac{1}{4}c_5(m-2)-c_6(m-4)=0.\tag{3.6.j}$$

The Lemma then follows by solving the Equations (3.6.g) and (3.6.j) for c_3, c_4, and c_5. □

3.6.2 The formula for a_5

The following result is due to by Branson et. al. [87]; we give the formula without proof as it is somewhat lengthy and combinatorial.

Theorem 3.6.5 *Let D be an operator of Laplace type on V over M where M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}$ define mixed boundary conditions and let $F = f\cdot\mathrm{Id}$ be a scalar operator where $f \in C^\infty(M)$. Then:*

$$
\begin{aligned}
a_5(F,D,\mathcal{B}) = (4\pi)^{-(m-1)/2}\tfrac{1}{5760}\int_{\partial M}\mathrm{Tr}\Big\{&F\{360\chi E_{;mm} + 1440E_{;m}S\\
&+720\chi E^2 + 240\chi E_{:aa} + 240\chi\tau E + 48\chi\tau_{;ii} + 20\chi\tau^2\\
&-8\chi\rho_{ij}\rho_{ij} + 8\chi R_{ijkl}R_{ijkl} - 120\chi\rho_{mm}E - 20\chi\rho_{mm}\tau\\
&+480\tau S^2 + 12\chi\tau_{;mm} + 24\chi\rho_{mm:aa} + 15\chi\rho_{mm;mm}\\
&+270\tau_{;m}S + 120\rho_{mm}S^2 + 960SS_{:aa} + 16\chi R_{ammb}\rho_{ab}\\
&-17\chi\rho_{mm}\rho_{mm} - 10\chi R_{ammb}R_{ammb} + 2880ES^2 + 1440S^4\\
&+(90\Pi_+ + 450\Pi_-)L_{aa}E_{;m} + (\tfrac{111}{2}\Pi_+ + 42\Pi_-)L_{aa}\tau_{;m}\\
&+30\Pi_+L_{ab}R_{ammb;m} + 240L_{aa}S_{:bb} + 420L_{ab}S_{:ab} + 390L_{aa:b}S_{:b}\\
&+480L_{ab:a}S_{:b} + 420L_{aa:bb}S + 60L_{ab:ab}S\\
&+(\tfrac{487}{16}\Pi_+ + \tfrac{413}{16}\Pi_-)L_{aa:b}L_{cc:b} + (238\Pi_+ - 58\Pi_-)L_{ab:a}L_{cc:b}\\
&+(\tfrac{49}{4}\Pi_+ + \tfrac{11}{4}\Pi_-)L_{ab:a}L_{bc:c} + (\tfrac{535}{8}\Pi_+ - \tfrac{355}{8}\Pi_-)L_{ab:c}L_{ab:c}\\
&+(\tfrac{151}{4}\Pi_+ + \tfrac{29}{4}\Pi_-)L_{ab:c}L_{ac:b} + (111\Pi_+ - 6\Pi_-)L_{aa:bb}L_{cc}\\
&+(-15\Pi_+ + 30\Pi_-)L_{ab:ab}L_{cc} + (-\tfrac{15}{2}\Pi_+ + \tfrac{75}{2}\Pi_-)L_{ab:ac}L_{bc}\\
&+(\tfrac{945}{4}\Pi_+ - \tfrac{285}{4}\Pi_-)L_{aa:bc}L_{bc} + (114\Pi_+ - 54\Pi_-)L_{bc:aa}L_{bc}\\
&+1440L_{aa}SE + 30L_{aa}S\rho_{mm} + 240L_{aa}S\tau - 60L_{ab}\rho_{ab}S\\
&+180L_{ab}SR_{ammb} + (195\Pi_+ - 105\Pi_-)L_{aa}L_{bb}E\\
&+(30\Pi_+ + 150\Pi_-)L_{ab}L_{ab}E + (\tfrac{195}{6}\Pi_+ - \tfrac{105}{6}\Pi_-)L_{aa}L_{bb}\tau\\
&+(5\Pi_+ + 25\Pi_-)L_{ab}L_{ab}\tau + (-\tfrac{275}{16}\Pi_+ + \tfrac{215}{16}\Pi_-)L_{aa}L_{bb}\rho_{mm}\\
&+(-\tfrac{275}{8}\Pi_+ + \tfrac{215}{8}\Pi_-)L_{ab}L_{ab}\rho_{mm} + (-\Pi_+ - 14\Pi_-)L_{cc}L_{ab}\rho_{ab}\\
&+(\tfrac{109}{4}\Pi_+ - \tfrac{49}{4}\Pi_-)L_{cc}L_{ab}R_{ammb} + 16\chi L_{ab}L_{ac}\rho_{bc}\\
&+(\tfrac{133}{2}\Pi_+ + \tfrac{47}{2}\Pi_-)L_{ab}L_{ac}R_{bmmc} - 32\chi L_{ab}L_{cd}R_{acbd}\\
&+\tfrac{315}{2}L_{cc}L_{ab}L_{ab}S + (\tfrac{2041}{128}\Pi_+ + \tfrac{65}{128}\Pi_-)L_{aa}L_{bb}L_{cc}L_{dd}\\
&+150L_{ab}L_{bc}L_{ac}S + (\tfrac{417}{32}\Pi_+ + \tfrac{141}{32}\Pi_-)L_{cc}L_{dd}L_{ab}L_{ab}\\
&+1080L_{aa}L_{bb}S^2 + 360L_{ab}L_{ab}S^2 + (\tfrac{375}{32}\Pi_+ - \tfrac{777}{32}\Pi_-)L_{ab}L_{ab}L_{cd}L_{cd}\\
&+\tfrac{885}{4}L_{aa}L_{bb}L_{cc}S + (25\Pi_+ - \tfrac{17}{2}\Pi_-)L_{dd}L_{ab}L_{bc}L_{ac}\\
&+2160L_{aa}S^3 + (\tfrac{231}{8}\Pi_+ + \tfrac{327}{8}\Pi_-)L_{ab}L_{bc}L_{cd}L_{da}\\
&-180E^2 + 180\chi E\chi E - 120S_{:a}S_{:a} + 720\chi S_{:a}S_{:a}\\
&-\tfrac{105}{4}\Omega_{ab}\Omega_{ab} + 120\chi\Omega_{ab}\Omega_{ab} + \tfrac{105}{4}\chi\Omega_{ab}\chi\Omega_{ab} - 45\Omega_{am}\Omega_{am}\\
&+180\chi\Omega_{am}\Omega_{am} - 45\chi\Omega_{am}\chi\Omega_{am} + 360(\Omega_{am}\chi S_{:a} - \Omega_{am}S_{:a}\chi)\\
&+45\chi\chi_{:a}\Omega_{am}L_{cc} - 180\chi_{:a}\chi_{:b}\Omega_{ab} + 90\chi\chi_{:a}\chi_{:b}\Omega_{ab}\\
&+90\chi\chi_{:a}\Omega_{am;m} + 120\chi\chi_{:a}\Omega_{ab:b} + 180\chi\chi_{:a}\Omega_{bm}L_{ab} + 300\chi_{:a}E_{:a}\\
&-180\chi_{:a}\chi_{:a}E - 90\chi\chi_{:a}\chi_{:a}E + 240\chi_{:aa}E - 30\chi_{:a}\chi_{:a}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\quad -60\chi_{:a}\chi_{:b}\rho_{ab} + 30\chi_{:a}\chi_{:b}R_{mabm}\\
&\quad -\tfrac{675}{32}\chi_{:a}\chi_{:a}L_{bb}L_{cc} - \tfrac{75}{4}\chi_{:a}\chi_{:b}L_{ac}L_{bc} - \tfrac{195}{16}\chi_{:a}\chi_{:a}L_{cd}L_{cd}\\
&\quad -\tfrac{675}{8}\chi_{:a}\chi_{:b}L_{ab}L_{cc} - 330\chi_{:a}S_{:a}L_{cc} - 300\chi_{:a}S_{:b}L_{ab}\\
&\quad +\tfrac{15}{4}\chi_{:a}\chi_{:a}\chi_{:b}\chi_{:b} + \tfrac{15}{8}\chi_{:a}\chi_{:b}\chi_{:a}\chi_{:b} - \tfrac{15}{4}\chi_{:aa}\chi_{:bb} - \tfrac{105}{2}\chi_{:ab}\chi_{:ab}\\
&\quad -15\chi_{:a}\chi_{:a}\chi_{:bb} - \tfrac{135}{2}\chi_{:b}\chi_{:aab}\}\\
&+F_{;m}\{(\tfrac{195}{2}\Pi_+ - 60\Pi_-)\tau_{;m} + 240\tau S - 90\rho_{mm}S + 270S_{:aa}\\
&\quad +(630\Pi_+ - 450\Pi_-)E_{;m} + 1440ES + 720S^3\\
&\quad +(90\Pi_+ + 450\Pi_-)L_{aa}E + (-\tfrac{165}{8}\Pi_+ - \tfrac{255}{8}\Pi_-)L_{aa}\rho_{mm}\\
&\quad +(15\Pi_+ + 75\Pi_-)L_{aa}\tau + 600L_{aa}S^2 + (\tfrac{1215}{8}\Pi_+ - \tfrac{315}{8}\Pi_-)L_{aa:bb}\\
&\quad -\tfrac{45}{4}\chi L_{ab:ab} + (15\Pi_+ - 30\Pi_-)L_{ab}\rho_{ab}\\
&\quad +(-\tfrac{165}{4}\Pi_+ + \tfrac{465}{4}\Pi_-)L_{ab}R_{ammb} + \tfrac{705}{4}L_{aa}L_{bb}S - \tfrac{75}{2}L_{ab}L_{ab}S\\
&\quad +(\tfrac{459}{32}\Pi_+ + \tfrac{495}{32}\Pi_-)L_{aa}L_{bb}L_{cc} + (\tfrac{267}{16}\Pi_+ - \tfrac{1485}{16}\Pi_-)L_{cc}L_{ab}L_{ab}\\
&\quad +(-54\Pi_+ + \tfrac{225}{2}\Pi_-)L_{ab}L_{bc}L_{ac}\\
&\quad -210\chi_{:a}S_{:a} - \tfrac{165}{16}\chi_{:a}\chi_{:a}L_{cc} - \tfrac{405}{8}\chi_{:a}\chi_{:b}L_{ab} + 135\chi\chi_{:a}\Omega_{am}\}\\
&+F_{;mm}\{30L_{aa}S + (\tfrac{315}{16}\Pi_+ - \tfrac{1215}{16}\Pi_-)L_{aa}L_{bb}\\
&\quad +(-\tfrac{645}{8}\Pi_+ + \tfrac{945}{8}\Pi_-)L_{ab}L_{ab}\\
&\quad +60\chi\tau - 90\chi\rho_{mm} + 360\chi E + 360S^2 - 30\chi_{:a}\chi_{:a}\}\\
&+F_{;mmm}\{180S + (-30\Pi_+ + 105\Pi_-)L_{aa}\} + 45\chi F_{;mmmm}\Big\}dy\,.
\end{aligned}
$$

3.7 Spectral geometry

Let M be a compact Riemannian manifold. Let $\operatorname{Spec}^p(M,\mathcal{B})$ be the spectrum of the p form valued Laplacian Δ^p with either Dirichlet ($\mathcal{B}=\mathcal{B}_D$) or Neumann ($\mathcal{B}=\mathcal{B}_N$) boundary conditions; we repeat each eigenvalue according to its multiplicity. If the boundary of M is empty, we shall omit $\mathcal{B}$ from the notation. Since the trace of the fundamental solution of the heat equation is determined by the spectrum, the heat trace asymptotics $a_n(1,\Delta^p,\mathcal{B})$ are spectral invariants. Let $c_{\mathcal{B}_D} := -\frac14$ and $c_{\mathcal{B}_N} := \frac14$. By Theorem 3.3.1,

$$\operatorname{Tr}_{L^2}\{e^{-t\Delta^p_{\mathcal{B}}}\} = (4\pi t)^{-m/2}\binom{m}{p}\{\operatorname{vol}(M) + c_{\mathcal{B}}\operatorname{vol}(\partial M)t^{1/2} + O(t)\}\,.$$

Consequently the dimension, volume of M, and volume of ∂M are spectral invariants. In this section, we will explore other relationships between the spectrum of the Laplacian and the geometry of the underlying manifold.

To simplify the formulae involved, we adopt the usual conventions defining the binomial coefficient that $0! = 1$ and that

$$\binom{a}{b} := \begin{cases} \frac{a!}{b!(a-b)!} & \text{if } 0 \le b \le a,\\ 0 & \text{otherwise}. \end{cases}$$

The following result follows from work of Patodi [301] and generalizes earlier work of Kac [250] and of McKean and Singer [278]. It is central to this subject.

Theorem 3.7.1 *Let M be a closed Riemannian manifold. Let $f \in C^\infty(M)$.*

1. $a_0(f, \Delta^p) = (4\pi)^{-m/2}\binom{m}{p}\operatorname{vol}(M)$.
2. $a_2(f, \Delta^p) = \frac{1}{6}(4\pi)^{-m/2}\int_M(\binom{m}{p} - 6\binom{m-2}{p-1})\tau dx$.
3. $a_4(f, \Delta^p) = \frac{1}{360}(4\pi)^{-m/2}\int_M\{(12\binom{m}{p} - 60\binom{m-2}{p-1})\tau_{;ii}$
$+(5\binom{m}{p} - 60\binom{m-2}{p-1} + 180\binom{m-4}{p-2})\tau^2$
$+(-2\binom{m}{p} + 180\binom{m-2}{p-1} - 720\binom{m-4}{p-2})|\rho|^2$
$+(2\binom{m}{p} - 30\binom{m-2}{p-1} + 180\binom{m-4}{p-2})|R|^2\}dx$.

We say that M has *constant sectional curvature* c if and only if

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\,. \tag{3.7.a}$$

For $r > 0$, let $\mathbb{H}^m(r) := \{x \in \mathbb{R}^m : x_m > 0\}$ with the hyperbolic metric

$$ds^2_{\mathbb{H}(r)} := r^2\frac{dx_1^2 + \ldots + dx_m^2}{x_m^2}$$

and let $S^m(r)$ be the sphere of radius r in $\mathbb{R}^{m+1}$. Then $\mathbb{H}^m(r)$ has constant sectional curvature $-r^{-2}$, $\mathbb{R}^m$ has constant sectional curvature 0, and $S^m(r)$ has constant sectional curvature r^{-2}. These model spaces are important because any two m dimensional manifolds with constant sectional curvature c are locally isometric. Thus, in particular, if M is a closed connected m dimensional Riemannian manifold with constant sectional curvature $+1$ and with $\operatorname{vol}(M) = \operatorname{vol}(S^m)$, then M is isometric to S^m. The following theorem, due to Berger [58] and Tanno [347], shows that standard spheres in low dimensions are characterized by their spectrum:

Theorem 3.7.2 *Let M_1 and M_2 be closed Riemannian manifolds of dimension $m \le 6$. Assume that $\operatorname{Spec}^0(M_1) = \operatorname{Spec}^0(M_2)$. If M_1 has constant sectional curvature c, then so does M_2.*

By considering more than one operator, it is possible to garner additional results of this form. Recall (M, g) is said to be *Einstein* if there is $\lambda \in \mathbb{R}$ so

$$\rho_{ij} = \lambda\delta_{ij}\,.$$

Patodi [301] showed:

Theorem 3.7.3 *Let M_1 and M_2 be closed Riemannian manifolds so that $\operatorname{Spec}^p(M_1) = \operatorname{Spec}^p(M_2)$ for $p = 0, 1, 2$. Then:*

1. *If M_1 has constant scalar curvature s, then so does M_2.*
2. *If M_1 is Einstein, then so is M_2.*
3. *If M_1 has constant sectional curvature c, then so does M_2.*

Park [297] generalized Patodi's result to the context of manifolds with boundary under the additional hypothesis that the manifolds in question had constant scalar curvature.

Theorem 3.7.4 *Let $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_N$. Let M_1 and M_2 be compact Riemannian manifolds with smooth boundaries with the same constant scalar curvature s. Assume* $\operatorname{Spec}^p(M_1, \mathcal{B}) = \operatorname{Spec}^p(M_2, \mathcal{B})$ *for $p = 0, 1, 2$. Then:*

1. *If M_1 is Einstein, then so is M_2.*
2. *If M_1 has constant sectional curvature c, then so does M_2.*

Let M be a compact Riemannian manifold with smooth boundary. We say that the boundary is *totally geodesic* if the second fundamental form vanishes identically on M; geodesics in M which are tangent to ∂M at a single point stay in ∂M for all time. We say that the boundary is *minimal* if the normalized mean curvature L_{aa} vanishes. We say that the boundary is *totally umbillic* if the second fundamental form has only one eigenvalue. The boundary is said to be *strongly totally umbillic* if the eigenvalue in question is constant and does not vary with the point in question. Park [298] showed the following result.

Theorem 3.7.5 *Let M_1 and M_2 be compact Einstein Riemannian manifolds with smooth boundaries and the same constant scalar curvature s. Assume that* $\operatorname{Spec}^0(M_1, \mathcal{B}_D) = \operatorname{Spec}^0(M_2, \mathcal{B}_D)$ *and* $\operatorname{Spec}^0(M_1, \mathcal{B}_N) = \operatorname{Spec}^0(M_2, \mathcal{B}_N)$.

1. *If M_1 has totally geodesic boundary, then so does M_2.*
2. *If M_1 has minimal boundary, then so does M_2.*
3. *If M_1 has totally umbillic boundary, then so does M_2.*
4. *If M_1 has strongly totally umbillic boundary, then so does M_2.*

The remainder of this section is devoted to the proof of these results. We have not tried to present the most general possible results in this genre and have contented ourselves with results which exemplify the general phenomena and which are easy to state and to prove.

3.7.1 Proof of Theorem 3.7.1

Let M be a closed Riemannian manifold of dimension m. Assertion (1) follows from Theorem 3.3.1 since

$$\operatorname{Tr}(\mathrm{Id}) = \dim \Lambda^p(M) = \binom{m}{p}.$$

If $m = 1$, then the remaining assertions are immediate since a_n vanishes for $n > 0$ and since $\tau = 0$, $\rho = 0$, and $R = 0$. We therefore suppose $m \geq 2$.

Since Δ^p is a natural differential operator in the context of Riemannian geometry, the integrand $a_2(x, \Delta^p)$ can be expressed in terms of τ and $a_4(x, \Delta^p)$ can be expressed in terms of $\{\tau_{;kk}, \tau^2, |\rho|^2, |R|^2\}$. Therefore, there exist universal constants so that

$$a_2(f, \Delta^p) = \tfrac{1}{6}(4\pi)^{-m/2} \int_M f c_{m,p} \tau dx, \tag{3.7.b}$$

$$a_4(f, \Delta^p) = \tfrac{1}{360}(4\pi)^{-m/2} \int_M f \Big\{ c^1_{m,p} \tau_{;ii} + c^2_{m,p} \tau^2 + c^3_{m,p} |\rho|^2 + c^4_{m,p} |R|^2 \Big\} dx .$$

The coefficient $c_{m,p}$ is uniquely specified if $m \geq 2$; the coefficients $c^i_{m,p}$ are uniquely specified if $m \geq 4$. The cases $m = 2$ and $m = 3$ are slightly exceptional. We have

$$\begin{array}{lll} |\rho|^2 = \frac{1}{2}\tau^2 \quad \text{and} \quad |R|^2 = \tau^2 & \text{if } m = 2, \\ |R|^2 = -\tau^2 + 4|\rho|^2 & \text{if } m = 3\,. \end{array} \tag{3.7.c}$$

Let $M = M_1 \times S^1$. Since the structures on S^1 are flat, the heat trace asymptotics on S^1 vanish for $n > 0$. Thus by Lemma 1.8.2, we have the recursion relation

$$c^i_{m,p} = c^i_{m-1,p} + c^i_{m-1,p-1}\,. \tag{3.7.d}$$

We apply Lemma 1.8.1 with $\phi = 0$ to see

$$c^i_{m,p} = c^i_{m,m-p}\,. \tag{3.7.e}$$

We use Theorem 3.3.1 to see that

$$\begin{aligned} a_2(f,\Delta^p) &= (4\pi)^{-m/2}\tfrac{1}{6}\int_M f\Big\{\tbinom{m}{p}\tau + \operatorname{Tr}_{\Lambda^p}(6E)\Big\}dx, \\ a_4(f,\Delta^p) &= (4\pi)^{-m/2}\tfrac{1}{360}\int_M f\Big\{\tbinom{m}{p}(12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2) \\ &+ \operatorname{Tr}_{\Lambda^p}(60E_{;kk} + 60\tau E + 180E^2 + 30\Omega_{ij}\Omega_{ij})\Big\}dx\,. \end{aligned}$$

If $p = 0$, then $E = 0$ and $\Omega = 0$ so Assertions (2) and (3) follow in this special case from these two equations. If $p = 1$, then we use Lemma 1.2.5 to see

$$Ee_k = -\tfrac{1}{2}\gamma_i\gamma_j\Omega_{ij}e_k = -\rho_{ik}e_i\,.$$

Consequently,

$$\begin{array}{ll} \operatorname{Tr}_{\Lambda^1}\{E\} = -\tau, & 60\operatorname{Tr}_{\Lambda^1}\{E_{;kk}\} = -60\tau_{;kk}, \\ 60\tau\operatorname{Tr}_{\Lambda^1}\{E\} = -60\tau^2, & 180\operatorname{Tr}_{\Lambda^1}\{E^2\} = 180|\rho|^2, \\ 30\operatorname{Tr}_{\Lambda^1}\{\Omega_{ij}\Omega_{ij}\} = -30|R|^2\,. & \end{array}$$

Assertions (2) and (3) now follow for $p = 1$ and general m. Assertions (2) and (3) for $(m,p) \in \{(2,2),(2,3),(3,3)\}$ follow from the cases $p = 0$ and $p = 1$ using the duality relation given in Equation (3.7.e). Assertion (2) for general (m,p) follows, by induction, from the recursion relation given in Equation (3.7.d) from the case $m = 2$ using Pascal's relation

$$\tbinom{k-1}{q-1} + \tbinom{k-1}{q} = \tbinom{k}{q}\,.$$

Let $m = 4$. Assertion (3) for $p = 3$ and $p = 4$ follows from Equation (3.7.e). Thus only the case $p = 2$ needs to be established. We use Theorem 1.9.1 to compute the supertrace

$$a^{d+\delta}_{4,4} = \tfrac{1}{\pi^2 8^2 2!}\varepsilon^I_J\mathcal{R}^{I,4}_{J,1} = (4\pi)^{-2}\tfrac{1}{8}\{4\tau^2 - 16|\rho|^2 + 4|R|^2\}\,.$$

We use this relation to determine the coefficients of $\binom{m-4}{p-2}$ appearing in

$a_2(x,\Delta^2)$; the terms involving $\binom{m}{p}$ and $\binom{m-2}{p-1}$ cancel in taking the supertrace. This establishes Assertion (3) for $m=4$ and $p=2$. This completes the proof if $m=4$. We use the recursion relation given in Equation (3.7.e) to complete the proof for general m. □

3.7.2 Proof of Theorem 3.7.2

Fix the constant c. Let ε be an auxiliary real parameter. Define the reduced scalar curvature $\tilde\tau$, the reduced Ricci tensor $\tilde\rho$, and the reduced Weyl curvature tensor $\tilde W$ by setting

$$\begin{aligned}
\tilde\tau : &= \tau - m(m-1)c, \\
\tilde\rho_{ij} : &= \rho_{ij} - (m-1)c\delta_{ij}, \quad \text{and} \\
\tilde W_{ijkl} : &= R_{ijkl} - \varepsilon(\tilde\rho_{il}\delta_{jk} + \tilde\rho_{jk}\delta_{il} - \tilde\rho_{ik}\delta_{jl} - \tilde\rho_{jl}\delta_{ik}) \\
&- c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) .
\end{aligned}$$

If M has constant sectional curvature c, then $\tilde\rho=0$ and $\tilde\tau=0$; hence $\tilde W=0$ as well. Conversely, if $\tilde\rho=0$ and $\tilde W=0$, then M has constant sectional curvature c. We may express

$$\begin{aligned}
|\tilde W|^2 &= |R|^2 - 2\varepsilon\{\tilde\rho_{il}R_{ijjl} + \tilde\rho_{jk}R_{ijki} - \tilde\rho_{ik}R_{ijkj} - \tilde\rho_{jl}R_{ijil}\} \\
&+ \varepsilon^2\{\tilde\rho_{il}\tilde\rho_{il}\delta_{jk}\delta_{jk} + \tilde\rho_{jk}\tilde\rho_{jk}\delta_{il}\delta_{il} + \tilde\rho_{ik}\tilde\rho_{ik}\delta_{jl}\delta_{jl} + \tilde\rho_{jl}\tilde\rho_{jl}\delta_{ik}\delta_{ik} \\
&+ 2\tilde\rho_{il}\tilde\rho_{jk}\delta_{jk}\delta_{il} - 2\tilde\rho_{il}\tilde\rho_{ik}\delta_{jk}\delta_{jl} - 2\tilde\rho_{il}\tilde\rho_{jl}\delta_{jk}\delta_{ik} - 2\tilde\rho_{jk}\tilde\rho_{ik}\delta_{il}\delta_{jl} \\
&- 2\tilde\rho_{jk}\tilde\rho_{jl}\delta_{il}\delta_{ik} + 2\tilde\rho_{ik}\tilde\rho_{jl}\delta_{jl}\delta_{ik}\} + \star(m,\varepsilon,c)\tau + \star(m,\varepsilon,c) \\
&= |R|^2 - 8\varepsilon|\tilde\rho|^2 + \varepsilon^2\{4m|\tilde\rho|^2 + 4\tau^2 - 8|\tilde\rho|^2\} + \star(m,\varepsilon,c)\tau + \star(m,\varepsilon,c)
\end{aligned}$$

where $\star(\cdot)$ denotes suitably chosen constants that are not of interest. Thus

$$|R|^2 = |\tilde W|^2 + (8\varepsilon + (8-4m)\varepsilon^2)|\tilde\rho|^2 - 4\varepsilon^2\tau^2 + \star(m,\varepsilon,c)\tau + \star(m,\varepsilon,c) .$$

Let M be a closed Riemannian manifold of dimension m. By Theorem 3.7.1,

$$\begin{aligned}
a_4(1,\Delta^0) &= \tfrac{1}{360}(4\pi)^{-m/2}\int_M \left\{5\tau^2 - 2\rho^2 + 2R^2\right\}dx \\
&= \tfrac{1}{360}(4\pi)^{-m/2}\int_M \left\{2|\tilde W|^2 + ((16-8m)\varepsilon^2 + 16\varepsilon - 2)|\tilde\rho|^2\right. \\
&+ \left.(5-8\varepsilon^2)\tau^2 + \star(m,\varepsilon,c)\tau + \star(m,\varepsilon,c)\right\}dx .
\end{aligned}$$

Since $a_0(1,\Delta^0) = (4\pi)^{-m/2}\int_M dx$ and $a_2(1,\Delta^0) = \frac{1}{6}(4\pi)^{-m/2}\int_M \tau dx$,

$$\begin{aligned}
&\int_M \left\{2|\tilde W|^2 + ((16-8m)\varepsilon^2 + 16\varepsilon - 2)|\tilde\rho|^2 + (5-8\varepsilon^2)\tilde\tau^2)\right\}dx \\
&= \star(m,\varepsilon,c)a_0(1,\Delta^0) + \star(m,\varepsilon,c)a_1(1,\Delta^0) + \star(m,\varepsilon,c)a_4(1,\Delta^0) .
\end{aligned}$$

If we can choose ε so that both the coefficients $\{(16-8m)\varepsilon^2 + 16\varepsilon - 2\}$ and $\{5-8\varepsilon^2\}$ are positive, then this spectral invariant will vanish if and only if M has constant sectional curvature c; this will complete the proof of the Theorem.

We set $\varepsilon = \frac{1}{3}$. We then have $5 - 8\varepsilon^2 > 0$. If $m \leq 5$, then

$$16\varepsilon + (16 - 8m)\varepsilon^2 - 2 \geq \tfrac{16}{3} - \tfrac{24}{9} - 2 = \tfrac{2}{3} > 0\,.$$

In the limiting case that $m = 6$, the optimal choice is $\varepsilon = \frac{1}{4}$ which yields

$$\int_{M_2} \left\{ 2|\tilde{W}|^2 + 3\tilde{\tau}^2 \right\} dx_2 = 0\,.$$

This implies that M has constant scalar curvature and that $\tilde{W} = 0$. The remaining difficulty is to show that $\tilde{\rho} = 0$. This rests upon an analysis of a_6 and we refer to Tanno [347] for further details. □

3.7.3 *Proof of Theorem 3.7.3*

We apply Theorem 3.7.1. It is immediate that $\int_M dx$ and $\int_M \tau dx$ are spectral invariants of Δ^0. Our first task is to show that the invariants

$$\int_M \tau^2 dx, \quad \int_M |\rho|^2 dx, \quad \text{and} \quad \int_M |R|^2 dx$$

are determined by the spectrum of the three operators Δ^0, Δ^1, and Δ^2.

The cases $m = 2$ and $m = 3$ are slightly exceptional. We use Display (3.7.c) to see that if $m = 2$, then

$$a_4(1, \Delta^0) = \tfrac{1}{360}(4\pi)^{-1} \int_M (5 - 1 + 2)\tau^2 dx$$

so $\int_M \tau^2 dx$ and hence $\int_M \rho^2 dx$ and $\int_M |R|^2 dx$ are spectral invariants. Furthermore, if $m = 3$, then

$$\begin{aligned}
a_4(1, \Delta^0) &= \tfrac{1}{360}(4\pi)^{-3/2} \int_M \left\{ (5-2)\tau^2 + (-2+8)|\rho|^2 \right\} dx, \\
a_4(1, \Delta^1) &= \tfrac{1}{360}(4\pi)^{-3/2} \int_M \Big\{ (3(5-2) - 60 + 30)\tau^2 \\
&\qquad\qquad + (3(-2+8) + 180 - 120)|\rho|^2 dx \Big\}\,.
\end{aligned}$$

Since the coefficient matrix

$$\begin{pmatrix} 3 & 6 \\ -21 & 78 \end{pmatrix}$$

is non-singular, the following invariants are spectrally determined if $m = 3$

$$\int_M \tau^2 dx, \quad \int_M |\rho^2| dx, \quad \int_M |R|^2 dx\,.$$

Finally, suppose $m \geq 4$. We adopt the notation of Equation (3.7.b) to define the constants $c^i_{m,p}$. We consider the coefficient matrix

$$\mathcal{A} := \begin{pmatrix} c^2_{m,0} & c^3_{m,0} & c^4_{m,0} \\ c^2_{m,1} & c^3_{m,1} & c^4_{m,1} \\ c^2_{m,2} & c^3_{m,2} & c^4_{m,2} \end{pmatrix}.$$

By Theorem 3.7.1,

$$\begin{aligned} \det \mathcal{A} &= \det \begin{pmatrix} 5 & -2 & 2 \\ -60 & 180 & -30 \\ 180 & -720 & 180 \end{pmatrix} \\ &= 30 \cdot 180 \cdot \det \begin{pmatrix} 5 & -2 & 2 \\ -2 & 6 & -1 \\ 1 & -4 & 1 \end{pmatrix} \\ &= 30 \cdot 180 \cdot \{5(6-4) + 2(-1) + 2(8-6)\} \\ &\neq 0\,. \end{aligned}$$

Thus in this case as well, $\int_M \tau^2 dx$, $\int_M |\rho|^2 dx$, and $\int_M |R|^2 dx$ are spectrally determined. We may therefore complete the proof of the Theorem by noting:

1. $\int_M (\tau - s)^2 dx = 0$ if and only if M has constant scalar curvature s.
2. $\int_M |\rho - \frac{s}{m} g|^2 dx = 0$ if and only if M is Einstein with scalar curvature s.
3. $\int_M |R_{ijkl} - c\delta_{il}\delta_{jk} + c\delta_{ik}\delta_{il}|^2 dx = 0$ if and only if M has constant sectional curvature c. □

3.7.4 Proof of Theorem 3.7.4

We use Theorems 3.4.1 and 3.5.1 with $S = 0$ throughout Section 3.7. Let $\mathcal{B}$ denote either Dirichlet or Neumann boundary conditions. To simplify the notation, we introduce the reduced invariants and the reduced coefficients

$$\begin{aligned} &\tilde{a}_n(1, \Delta^p, \mathcal{B}) := a_n(1, \Delta^p, \mathcal{B}) - \tbinom{m}{p} a_n(1, \Delta^0, \mathcal{B}), \quad \text{and} \\ &\tilde{c}^i_{m,p} := c^i_{m,p} - \tbinom{m}{p} c^i_{m,0}\,. \end{aligned}$$

Since the scalar curvature $\tau = s$ is a fixed constant, one uses $a_2(1, \Delta^0, \mathcal{B})$ to see that $\int_{\partial M} L_{aa} dy$ is a spectral invariant.

The scalar invariants involving the curvature tensor and the second fundamental form in the formulae for a_4 are multiplied by $\binom{m}{p}$ and therefore do not appear in the reduced invariant $\tilde{a}_4(1, \Delta^p, \mathcal{B})$. We showed in the proof of Theorem 3.7.1 that

$$\operatorname{Tr}(E \text{ on } \Lambda^p) = \tbinom{m-2}{p-1}\tau \quad \text{so} \quad \operatorname{Tr}(E_{;m} \text{ on } \Lambda^p) = \tau_{;m} = 0\,.$$

Thus the only boundary contribution appearing in $\tilde{a}_4(1, \Delta^p, \mathcal{B})$ is a multiple of the spectral invariant

$$\int_{\partial M} \tau L_{aa} dy = s \int_{\partial M} L_{aa} dy\,.$$

We wish to show $\int_M |\rho|^2 dx$ and $\int_M |R|^2 dx$ are also spectral invariants. Once this is shown, the remainder of the proof will then follow along exactly the same lines as those used to prove Theorem 3.7.3. This is immediate if $m = 2$ by Equation (3.7.c). If $m = 3$, we use Equation (3.7.c) to see

$$\tilde{a}_4(1, \Delta^1, \mathcal{B}) = \tfrac{1}{360}(4\pi)^{-2} \int_M \left\{ (180 - 120)|\rho|^2 + \star(m, s) \right\} dx$$

and hence $\int_M |\rho|^2 dx$ and $\int_M |R|^2 dx$ are spectral invariants. If $m \geq 4$, then let

$$\mathcal{A} := \begin{pmatrix} \tilde{c}^3_{m,1} & \tilde{c}^4_{m,1} \\ \tilde{c}^3_{m,2} & \tilde{c}^4_{m,2} \end{pmatrix} = \begin{pmatrix} 180 & -30 \\ 180(m-2) - 720 & -30(m-2) + 180 \end{pmatrix}.$$

Since $\det(\mathcal{A}) \neq 0$, we conclude that $\int_M |\rho|^2 dx$ and $\int_M |R|^2 dx$ are spectral invariants as desired. □

3.7.5 Proof of Theorem 3.7.5

Since the scalar curvature s is fixed, we may use $a_2(1, \Delta^0, \mathcal{B})$ to see $\int_{\partial M} L_{aa} dy$ is a spectral invariant. We set $E = 0$ in studying Δ^0. Since the manifolds are Einstein, $R_{amam} = -\frac{s}{m-1}$. As we are considering pure Neumann boundary conditions, we set $S = 0$. Thus Theorems 3.4.1 and 3.5.1 yield that

$$\int_{\partial M} \left\{ 7L_{aa}L_{bb} - 10L_{ab}L_{ab} \right\} dy \quad \text{and} \quad \int_{\partial M} \left\{ 13L_{aa}L_{bb} + 2L_{ab}L_{ab} \right\} dy$$

are spectral invariants. Since the coefficient matrix is non-singular, we see that

$$\int_{\partial M} L_{aa}L_{bb} dy \quad \text{and} \quad \int_{\partial M} L_{ab}L_{ab} dy$$

are spectral invariants individually. We may now complete the proof of Theorem 3.7.5. We argue that:

1. M has totally geodesic boundary if and only if $\int_{\partial M} L_{ab}L_{ab} dy = 0$. This is a spectral invariant.

2. M has minimal boundary if and only if $\int_{\partial M} L_{aa}L_{bb} dy = 0$. This is a spectral invariant.

3. M has strongly totally umbilic boundary with eigenvalue λ if and only if $\int_{\partial M} |L_{ab} - \lambda\delta_{ab}|^2 dy = 0$. This is a spectral invariant.

4. Let κ_a be the eigenvalues of the second fundamental form. Then M has totally umbillic boundary if and only if $\sum_{a<b}(\kappa_a - \kappa_b)^2 = 0$. Since

$$\begin{aligned} \sum_{a<b}(\kappa_a - \kappa_b)^2 &= (m-3)\sum_a \kappa_a^2 - \sum_a \kappa_a \sum_b \kappa_b \\ &= (m-3)L_{ab}L_{ab} - L_{aa}L_{bb}, \end{aligned}$$

this is spectrally determined. □

3.8 Supertrace asymptotics for the Witten Laplacian

Let (M, g) be a compact m dimensional manifold with smooth boundary ∂M. We adopt the notation of Section 1.2.6. Let ϕ be an auxiliary smooth function on M which is called the *dilaton*. We twist the exterior derivative d and the coderivative δ to define

$$d_\phi := e^{-\phi} d e^{\phi} \quad \text{and} \quad \delta_\phi := e^{\phi} \delta e^{-\phi} .$$

The associated *Witten Laplacian* is then given by

$$\Delta_\phi := d_\phi \delta_\phi + \delta_\phi d_\phi \quad \text{on} \quad C^\infty(\Lambda(M)) .$$

There are operators Δ_ϕ^p of Laplace type on $C^\infty(\Lambda^p(M))$ so that

$$\Delta_\phi = \oplus_p \Delta_\phi^p .$$

We adopt the notation of Section 1.8 to define the invariants

$$\begin{aligned} a_{n,m}^{d+\delta}(\phi, g)(x) \;&:= \sum_p (-1)^p a_n(x, \Delta_\phi^p), \quad \text{and} \\ a_{n,m,k}^{d+\delta}(\phi, g)(y) &:= \sum_p (-1)^p a_{n,k}(y, \Delta_\phi^p, \mathcal{B}_a) . \end{aligned}$$

If $\{e_i\}$ is a local orthonormal frame, let $\mathfrak{e}_i$ and $\mathfrak{i}_i$ denote exterior multiplication by e_i and the dual, interior multiplication by e_i. Let Ω_{ij} be the curvature of the Levi-Civita connection acting on the exterior algebra. We use Lemma 1.2.1 to express

$$\Delta_\phi = -(\operatorname{Tr}(\nabla^2) + E)$$

where, by Lemma 1.2.8, ∇ is the Levi-Civita connection and

$$E_{\phi,g} := -\tfrac{1}{2}\gamma_i\gamma_j\Omega_{ij} - \phi_{;i}\phi_{;i} - \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i) .$$

Thus the formalism of the previous sections can easily be employed to compute the heat trace asymptotics a_n and heat content asymptotics β_n for small n.

We summarize some vanishing results established previously in Section 1.8:

Theorem 3.8.1

1. *If m is odd and if $\phi = 0$, then $a_{n,m}^{d+\delta} = 0$.*
2. *If $n < m$ or if n is odd, then $a_{n,m}^{d+\delta} = 0$.*
3. *If $n + k < m$, then $a_{n,m,k}^{d+\delta} = 0$.*

Let ε_I^J be the totally anti-symmetric tensor defined in Equation (1.7.i). If I and J are m tuples of indices indexing an orthonormal frame for the tangent bundle of M, set

$$\mathcal{R}_{J,s}^{I,t} := R_{i_s i_{s+1} j_{s+1} j_s} \ldots R_{i_{t-1} i_t j_t j_{t-1}} .$$

The following results, which generalize earlier results of Gilkey [181] for the special case $\phi = 0$, are due to Gilkey, Kirsten, and Vassilevich [202, 203] and to Gilkey, Kirsten, Vassilevich, and Zelnikov [204]. They deal with the first non-vanishing interior heat super-trace asymptotics:

Theorem 3.8.2 *We have*

1. $a^{d+\delta}_{2\bar m,2\bar m} = (\pi^{\bar m}8^{\bar m}\bar m!)^{-1}\varepsilon^I_J\mathcal{R}^{I,m}_{J,1}$.
2. $a^{d+\delta}_{2\bar m+2,2\bar m+1} = (\sqrt{\pi}\pi^{\bar m}8^{\bar m}\bar m!)^{-1}\varepsilon^I_J\phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}$.
3. *Let* $m = 2$. *Then* $a^{d+\delta}_{4,2} = (2\pi)^{-1}\varepsilon^I_J(\phi_{;i_1j_1}\phi_{;i_2})_{;j_2} + (24\pi)^{-1}R_{ijji;kk}$.
4. *Let* $m = 2\bar m \ge 4$. *Then* $a^{d+\delta}_{2\bar m+2,2\bar m} = (12\pi^{\bar m}8^{\bar m}\bar m!)^{-1}\varepsilon^I_J(\mathcal{R}^{I,m}_{J,1})_{;kk}$
   $+4(\pi^{\bar m}8^{\bar m}(\bar m-1)!)^{-1}\varepsilon^I_J(\phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3})_{;j_2}$
   $+(6\pi^{\bar m}8^{\bar m}(\bar m-1)!)^{-1}(R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3})_{;j_2}$.

Next we study the boundary terms. Let

$$\mathcal{R}^{A,t}_{B,s} := R_{a_sa_{s+1}b_{s+1}b_s}...R_{a_{t-1}a_tb_tb_{t-1}} \quad \text{and} \quad \mathcal{L}^{A,t}_{B,s} := L_{a_sb_s}...L_{a_tb_t} .$$

We introduced the following invariants previously in Display (1.8.l), setting

$$\begin{aligned}
\mathcal{F}^k_{m-1,m} &:= \varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1},\\
\mathcal{F}^{1,k}_{m,m} &:= \varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\phi_{;a_{2k+1}b_{2k+1}}\mathcal{L}^{A,m-1}_{B,2k+2},\\
\mathcal{F}^{2,k}_{m,m} &:= \varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\phi_{;a_{2k+1}}\phi_{;b_{2k+1}}\mathcal{L}^{A,m-1}_{B,2k+2},\\
\mathcal{F}^{3,k}_{m,m} &:= \varepsilon^A_B\{\mathcal{R}^{A,2k}_{B,1}R_{a_{2k+1}a_{2k+2}b_{2k+2}m}\mathcal{L}^{A,m-1}_{B,2k+3}\}_{:b_{2k+1}} .
\end{aligned}$$

By Lemma 1.8.11, there exist universal constants so that

$$\begin{aligned}
&a^{d+\delta}_{m,m,0} = \sum_k c^k_{m,m,0}\mathcal{F}^k_{m-1,m}, \quad a^{d+\delta}_{m+1,m,0} = \sum_{k,i} c^{i,k}_{m+1,m,0}\mathcal{F}^{i,k}_{m,m},\\
&a^{d+\delta}_{m+1,m,1} = \sum_k c^k_{m+1,m,1}\mathcal{F}^k_{m-1,m} . \qquad (3.8.a)
\end{aligned}$$

The following result corrects a minor error of [202] in the computation of the coefficient $c^{3,k}_{m+1,m,0}$.

Theorem 3.8.3 *Adopt the notation established above. Then*

1. $c^k_{m,m,0} = \frac{1}{\pi^k 8^k k!\operatorname{vol}(S^{m-2k-1})(m-2k-1)!}$.
2. $c^k_{m+1,m,1} = \frac{\sqrt{\pi}}{8^k\pi^k k!\operatorname{vol}(S^{m-2k})(m-2k)!}$.
3. $c^{1,k}_{m+1,m,0} = \frac{1}{\sqrt{\pi}8^k\pi^k k!\operatorname{vol}(S^{m-2k-2})(m-2k)!}$.
4. $c^{2,k}_{m+1,m,0} = 0$.
5. $c^{3,k}_{m+1,m,0} = \frac{1}{4\sqrt{\pi}8^k\pi^k k!\operatorname{vol}(S^{m-2k-2})(m-2k-2)!}$.

We introduce constants that describe the value of certain invariants on the standard sphere (S^m, g_{S^m}) and the standard disk (D^m, g_{D^m}). Let

$$\begin{aligned}
\mathfrak{R}_m &:= \varepsilon^I_J\mathcal{R}^{I,m}_{J,1}(g_{S^m}) = 4^{\bar m}\bar m! \quad \text{for} \quad m = 2\bar m,\\
\mathfrak{L}_m &:= \varepsilon^B_A\mathcal{L}^{B,m-1}_{A,1}(g_{D^m}) = (m-1)! .
\end{aligned}$$

3.8.1 The proof of Theorem 3.8.2

Let m be even. By Lemma 1.8.9, there is a universal constant so that

$$a^{d+\delta}_{m,m}(\phi,g)=c_{m,m}\varepsilon^I_J\mathcal{R}^{I,m}_{J,1}\,.$$

Thus ϕ plays no role in $a^{d+\delta}_{m,m}$ so Assertion (1) of Theorem 3.8.2 follows from Theorem 1.9.1.

Before establishing the remaining assertions of Theorem 3.8.2, we must first establish a normalizing constant in the 1 dimensional setting:

Lemma 3.8.4 *If $M=S^1$, then $a^{d+\delta}_{2,1}(\phi,g)=\frac{1}{\sqrt{\pi}}\phi_{;11}$.*

Proof: Let $M=S^1$. By Lemma 1.2.8,

$$E_\phi=-\phi_{;i}\phi_{;i}-\phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j-\mathfrak{i}_j\mathfrak{e}_i)\,.$$

Consequently by Theorem 3.3.1 we have

$$\begin{aligned}
&a_2(x,\Delta^0_\phi)=(4\pi)^{-1/2}\mathrm{Tr}\,(E_0)=(4\pi)^{-1/2}\{-\phi^2_{;1}+\phi_{;11}\},\\
&a_2(x,\Delta^1_\phi)=(4\pi)^{-1/2}\mathrm{Tr}\,(E_1)=(4\pi)^{-1/2}\{-\phi^2_{;1}-\phi_{;11}\},\\
&a^{d+\delta}_{2,1}=2(4\pi)^{-1/2}\phi_{;11}\,.\qquad\Box
\end{aligned}$$

Suppose that $m=2\bar m+1$ is odd. By Lemma 1.8.9, there is a universal constant $c_{m+1,m}$ so that

$$a^{d+\delta}_{m+1,m}(\phi,g)=c_{m+1,m}\varepsilon^I_J\phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}\,.$$

Give $M:=S^1\times S^{2\bar m}$ the product structures. The product and vanishing results established previously then yield

$$a^{d+\delta}_{m+1,m}(\phi_M,g_M)=a^{d+\delta}_{2,1}(\phi_1,g_1)a^{d+\delta}_{2\bar m,2\bar m}(\phi_2,g_2)\,. \tag{3.8.b}$$

Let $\phi=\phi(\theta)$. We may use Theorem 1.9.1 to see

$$\begin{aligned}
&a^{d+\delta}_{m+1,m}(\phi,d\theta^2+g_{S^{2\bar m}})=c_{m+1,m}\mathfrak{R}_{m-1}\phi_{;11}\\
=\;&a^{d+\delta}_{2,1}(\phi,d\theta^2)\cdot a^{d+\delta}_{m-1,m-1}(0,g_{S^{2\bar m}})=\tfrac{1}{\sqrt{\pi}8^{\bar m}\pi^{\bar m}\bar m!}\mathfrak{R}_{m-1}\phi_{;11}\,.
\end{aligned}$$

Assertion (2) now follows as this relation implies that

$$c_{m+1,m}=\tfrac{1}{\sqrt{\pi}8^{\bar m}\pi^{\bar m}\bar m!}\,.$$

Assertions (3) and (4) can now be verified. Let $m=2\bar m$. By Lemma 1.8.9, there are universal constants so that

$$\begin{aligned}
a^{d+\delta}_{m+2,m}(\phi,g)=&\,c^1_{m+2,m}(\varepsilon^I_J\phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3})_{;j_2}\\
&+c^2_{m+2,m}(\varepsilon^I_J\mathcal{R}^{I,m}_{J,1})_{;kk}+c^3_{m+2,m}(\varepsilon^I_JR_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3})_{;j_2}\,.
\end{aligned}$$

Note that by the second Bianchi identity one has

$$\{\varepsilon^I_JR_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3}\}_{;j_2}=\varepsilon^I_JR_{i_1i_2kj_1;kj_2}\mathcal{R}^{I,m}_{J,3}\,. \tag{3.8.c}$$

Give $M := S^{m-2} \times S^1 \times S^1$ the product metric. Let $\phi = \phi_1(\theta_1) + \phi_2(\theta_2)$. One has

$$\begin{aligned}
a^{d+\delta}_{m+2,m}(\phi, g) &= 2c^1_{m+2,m}\mathfrak{R}_{m-2}\phi_{1;m-1m-1}\phi_{2;mm} \\
&= a^{d+\delta}_{m-2,m-2}(0, g_{S^{m-2}}) \cdot a^{d+\delta}_{2,1}(\phi_1, d\theta_1^2) \cdot a^{d+\delta}_{2,1}(\phi_2, d\theta_2^2) \\
&= \tfrac{1}{\pi^{\bar{m}-1}8^{\bar{m}-1}(\bar{m}-1)!} \cdot \tfrac{1}{\pi}\mathfrak{R}_{m-2}\phi_{1;m-1,m-1}\phi_{2;mm} \quad \text{so} \\
c^1_{m+2,m} &= \tfrac{4}{\pi^{\bar{m}}8^{\bar{m}}(\bar{m}-1)!} \,.
\end{aligned}$$

For the remainder of the proof we shall set $\phi = 0$ as it plays no further role. We first suppose $m = 2$; in this context the invariants

$$\{(\varepsilon^I_J\mathcal{R}^{I,m}_{J,1})_{;kk},\ (\varepsilon^I_J R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3})_{;j_2}\}$$

are not linearly independent. By Lemma 1.2.5, $E = -\frac{1}{2}\gamma_i\gamma_j\Omega_{ij}$. Thus

$$E = \left\{ \begin{array}{rl} 0 & \text{on } \Lambda^0 M, \\ -R_{1221}\mathrm{Id} & \text{on } \Lambda^1 M, \\ 0 & \text{on } \Lambda^2 M\,. \end{array} \right.$$

If $\mathcal{E}$ is a collection of endomorphisms $\mathcal{E}_p$ of $\Lambda^p M$, then the supertrace is defined by setting

$$\mathfrak{Tr}(\mathcal{E}) := \sum_p (-1)^p \mathrm{Tr}\,(\mathcal{E} \text{ on } \Lambda^p M)\,.$$

For example, $\mathfrak{Tr}(\mathrm{Id}\,) = 0$. We verify Assertion (3) by computing

$$a^{d+\delta}_{4,2}(g_N) = \tfrac{1}{4\pi}\tfrac{1}{6}\mathfrak{Tr}\{E\}_{;kk} + O(R^2) = \tfrac{1}{4\pi}\tfrac{1}{6}R_{ijji;kk} + O(R^2)\,.$$

Next we suppose that $m = 4$. By Theorems 3.3.1 and 3.8.1,

$$\begin{aligned}
0 = a^{d+\delta}_{0,4} &= \tfrac{1}{4^2\pi^2}\mathfrak{Tr}\{\mathrm{Id}\,\}, \\
0 = a^{d+\delta}_{2,4} &= \tfrac{1}{4^2\pi^2}\mathfrak{Tr}\{E\}, \\
a^{d+\delta}_{4,4} &= \tfrac{1}{4^2\pi^2}\mathfrak{Tr}\{\tfrac{1}{2}EE + \tfrac{1}{12}\Omega_{ij}\Omega_{ij}\}\,.
\end{aligned}$$

This permits us to express

$$\begin{aligned}
a^{d+\delta}_{6,4} = \tfrac{1}{4^2\pi^2}\mathfrak{Tr}\{&\tfrac{1}{45}\Omega_{ij;k}\Omega_{ij;k} + \tfrac{1}{180}\Omega_{ij;j}\Omega_{ik;k} + \tfrac{1}{60}\Omega_{ij;kk}\Omega_{ij} \\
&+ \tfrac{1}{60}\Omega_{ij}\Omega_{ij;kk} + \tfrac{1}{6}EE_{;ii} + \tfrac{1}{12}E_{;i}E_{;i}\} + O(R^3)\,. \qquad (3.8.\mathrm{d})
\end{aligned}$$

We study the expression $R_{1221}R_{3443}$ in $a^{d+\delta}_{4,4}$ and the expression $R_{1221;2}R_{3443;2}$ in $a^{d+\delta}_{6,4}$ and suppress other terms. Since $\mathrm{Tr}_{\Lambda^p}(\Omega_{ij}\Omega_{ij})$ does not give rise to $R_{1221}R_{3443}$, Assertion (1) implies

$$a^{d+\delta}_{4,4} = \tfrac{1}{2\cdot 4^2\pi^2}\mathfrak{Tr}\{E^2\} + \ldots = \tfrac{1}{8^2\pi^2 2!}\mathcal{R}^{I,4}_{J,1} = \tfrac{32}{8^2\pi^2 2!}R_{1221}R_{3443} + \ldots\,. \qquad (3.8.\mathrm{e})$$

Similarly only $\mathrm{Tr}_{\Lambda^p}(E_{;i}E_{;i})$ can give rise to the monomial $R_{1221;2}R_{3443;2}$. Consequently Equations (3.8.c), (3.8.d), and (3.8.e) yield

$$\begin{aligned}
a^{d+\delta}_{6,4} &= \tfrac{1}{4^2\pi^2}\tfrac{1}{12}\mathfrak{Tr}\{E_{;i}E_{;i}\} + \ldots = \tfrac{1}{4^2\pi^2}\tfrac{1}{24}\mathfrak{Tr}\{E^2\}_{;ii} + \ldots \\
= &\ \tfrac{1}{12}(a^{d+\delta}_{4,4})_{;kk} + \ldots = \tfrac{1}{6}\tfrac{32}{8^2\pi^2 2!}R_{1221;2}R_{3443;2} + \ldots
\end{aligned}$$

$$= \quad 2c_{6,4}^2 \varepsilon_J^I R_{i_1i_2j_2j_1;k} R_{i_3i_4j_4j_3;k} + \ldots = 64 c_{6,4}^2 R_{1221;2} R_{3443;2} + \ldots .$$

This shows that

$$c_{6,4}^2 = \tfrac{1}{12} \tfrac{1}{8^2\pi^2 2!} .$$

If $m = 2\bar{m} > 4$, let $(M, g) := (N^4 \times S^{m-4}, g_N + g_{S^{m-4}})$. Let

$$\mathcal{X}_4 := \varepsilon_{J,1}^{I,4} (R_{i_1i_2j_2j_1;k} R_{i_3i_4j_4j_3;k})(g_N) .$$

The invariants $a_{n,m}^{d+\delta}(0, g)$ are defined solely in terms of the metric. Thus they are invariant under isometric actions. The sphere S^{m-4} is a homogeneous space; the orthogonal group $O(m-3)$ acts transitively on S^{m-4}. Consequently, $a_{n,m}^{d+\delta}(g_{S^{m-4}})$ is constant. Since by Theorem 1.3.9 the integral vanishes for $n \neq m$, we have a pointwise vanishing. Consequently by Lemma 1.8.2,

$$\begin{aligned} & a_{m+2,m}^{d+\delta}(0, g) = \bar{m}(\bar{m}-1) c_{m+2,m}^2 \mathcal{X}_4 \mathfrak{R}_{m-4} + \ldots \\ = \quad & a_{6,4}^{d+\delta}(0, g_N) a_{m-4,m-4}^{d+\delta}(0, g_{S^{m-4}}) \\ = \quad & 2 \tfrac{1}{12} \tfrac{1}{8^2\pi^2 2!} \tfrac{1}{\pi^{\bar{m}-2} 8^{\bar{m}-2} (\bar{m}-2)!} \mathcal{X}_4 \mathfrak{R}_{m-4} . \end{aligned}$$

This implies that

$$c_{m+2,m}^2 \quad = \quad \tfrac{1}{12} \tfrac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} .$$

We complete the proof of Assertion (4) by determining $c_{m+2,m}^3$. Let

$$(M, g) = (N^2 \times S^{m-2}, g_N + g_{S^{m-2}}) .$$

Let $\mathcal{X}_2 := R_{ijji;kk}(g_N)$. Then making a similar argument and by applying Assertion (3), one has

$$\begin{aligned} & a_{m+2,m}^{d+\delta}(0, g) = (2\bar{m} c_{m+2,m}^2 + c_{m+2,m}^3) \mathcal{X}_2 \mathfrak{R}_{m-2} \\ = \quad & a_{4,2}^{d+\delta}(0, g_N) a_{m-2,m-2}^{d+\delta}(0, g_{S^{m-2}}) = \tfrac{1}{4\pi} \tfrac{1}{6} \tfrac{1}{\pi^{\bar{m}-1} 8^{\bar{m}-1} (\bar{m}-1)!} \mathcal{X}_2 \mathfrak{R}_{m-2} . \end{aligned}$$

This implies that

$$\begin{aligned} & 2\bar{m} c_{m+2,m}^2 + c_{m+2,m}^3 = \tfrac{1}{3\pi^{\bar{m}} 8^{\bar{m}} (\bar{m}-1)!} \quad \text{so} \\ & c_{m+2,m}^3 = \tfrac{1}{6\pi^{\bar{m}} 8^{\bar{m}} (\bar{m}-1)!} . \qquad \square \end{aligned}$$

3.8.2 The proof of Theorem 3.8.3

To establish Theorem 3.8.3, we must evaluate the universal constants of Equation (3.8.a). We break the proof into 3 different Lemmas. We will begin by using product formulae to show:

Lemma 3.8.5

1. If $k > 0$, then:

(a) $c_{m,m,0}^k = \frac{1}{\pi^k 8^k k!} c_{m-2k,m-2k,0}^0$,

(b) $c_{m+1,m,1}^k = \frac{1}{\pi^k 8^k k!} c_{m-2k+1,m-2k,1}^0$,

(c) $c^{i,k}_{m+1,m,0} = \frac{1}{\pi^k 8^k k!} c^{i,0}_{m-2k+1,m-2k,0}$.

2. *We have*

(a) $c^{0}_{m,m,0} = \frac{1}{\mathrm{vol}(\mathrm{S}^{\mathrm{m}-1})(\mathrm{m}-1)!}$,

(b) $c^{1,0}_{m+1,m,0} = \frac{1}{\sqrt{\pi}\mathrm{vol}(\mathrm{S}^{\mathrm{m}-2})(\mathrm{m}-2)!}$,

(c) $c^{2,0}_{m+1,m,0} = 0$.

Only the universal constants $c^{0}_{m+1,m,1}$ and $c^{3,0}_{m+1,m,0}$ remain to be determined. Introduce universal constants $\mathfrak{c}^{\nu}_{n,m,k}$ so that if $\mathcal{B}$ defines mixed boundary conditions for an operator D of Laplace type, then

$$\begin{aligned} a_{n,m,k}(y, D, \mathcal{B}) &= \mathfrak{c}^{0}_{n,m,k} \mathrm{Tr}\left\{S^{n-k-1}\right\} + \mathfrak{c}^{2}_{n,m,k} \mathrm{Tr}\left\{ES^{n-k-3}\right\} \\ &+ \mathfrak{c}^{3}_{n,m,k} \mathrm{Tr}\left\{E_{;m} S^{n-k-4}\right\} + \text{other terms}\,. \end{aligned}$$

Lemma 3.8.6

1. *If* $m \geq 2$, *then* $c^{0}_{m+1,m,1} = \mathfrak{c}^{0}_{m+1,m,1}$ *and* $c^{0}_{m,m,0} = \mathfrak{c}^{0}_{m,m,0}$.

2. *If* $m > 3$, *then* $c^{3,0}_{m+1,m,0} = \mathfrak{c}^{3}_{m+1,m,0}$.

We shall complete our computation of $a^{d+\delta}_{m,m}$ and $a^{d+\delta}_{m+1,m}$ by showing:

Lemma 3.8.7

1. $\mathfrak{c}^{i}_{n,m,k} = (4\pi)^{-(m-1)/2} \mathfrak{c}^{i}_{n,1,k}$.

2. *If* $n \geq 3$, *then* $\mathfrak{c}^{0}_{n,m,1} = \frac{1}{2} \mathfrak{c}^{0}_{n,m,0}$.

3. *If* $n \geq 5$, *then* $\mathfrak{c}^{3}_{n,m,0} = \mathfrak{c}^{0}_{n-2,m,1}$.

Proof of Lemma 3.8.5: Let $M := M_1 \times M_2$ be given product structures induced from the corresponding structures on the manifolds M_i. We suppose M_1 is a closed manifold. We then have

$$\begin{aligned} & a^{d+\delta}_{m,m,0}(\phi_M, g_M) = a^{d+\delta}_{m_1,m_1}(\phi_{M_1}, g_{M_1}) \cdot a^{d+\delta}_{m_2,m_2,0}(\phi_{M_2}, g_{M_2}), \\ & a^{d+\delta}_{m+1,m,1}(\phi_M, g_M) = a^{d+\delta}_{m_1,m_1}(\phi_{M_1}, g_{M_1}) \cdot a^{d+\delta}_{m_2+1,m_2,1}(\phi_{M_2}, g_{M_2}), \\ & a^{d+\delta}_{m+1,m,0}(\phi_M, g_M) = a^{d+\delta}_{m_1+1,m_1}(\phi_{M_1}, g_{M_1}) \cdot a^{d+\delta}_{m_2,m_2,0}(\phi_{M_2}, g_{M_2}) \\ & \qquad + a^{d+\delta}_{m_1,m_1}(\phi_{M_1}, g_{M_1}) \cdot a^{d+\delta}_{m_2+1,m_2,0}(\phi_{M_2}, g_{M_2})\,. \end{aligned} \tag{3.8.f}$$

Give $M = S^{2k} \times D^{m-2k}$ the natural product metric. By Theorem 3.8.2 and Equation (3.8.f),

$$\begin{aligned} & a^{d+\delta}_{m,m,0}(0, g_M) = c^{k}_{m,m,0} \mathcal{F}^{k}_{m-1,m}(0, g_M) = c^{k}_{m,m,0} \mathfrak{R}_{2k} \mathfrak{L}_{m-2k} \\ = \;& a^{d+\delta}_{2k,2k}(0, g_{S^{2k}}) \cdot a^{d+\delta}_{m-2k,m-2k,0}(0, g_{D^{m-2k}}) \\ = \;& \tfrac{1}{\pi^k 8^k k!} c^{0}_{m-2k,m-2k,0} \mathfrak{R}_{2k} \mathfrak{L}_{m-2k}\,. \end{aligned}$$

The first identity of Lemma 3.8.5 (1) now follows; the remaining two identities are proved similarly.

The curvature tensor of D^m vanishes. We derive the first identity of Assertion (2) by computing

$$\begin{aligned} 1 &= \chi(D^m) = \int_{\partial D^m} a^{d+\delta}_{m,m,0}(0, g_{D^m})dy \\ &= c^0_{m,m,0}\mathrm{vol}(S^{m-1})(m-1)!\,. \end{aligned}$$

Let $M := S^1 \times D^{m-1}$ where $\phi = \phi(\theta)$ depends only on S^1. By Lemma 3.8.4, $a^{d+\delta}_{2,1} = \frac{1}{\sqrt{\pi}}\phi_{;11}$. Consequently,

$$\begin{aligned} & a^{d+\delta}_{m+1,m,0}(\phi, g) = \{c^{1,0}_{m+1,m,0}\phi_{;11} + c^{2,0}_{m+1,m,0}\phi_{;1}\phi_{;1}\}\mathfrak{L}_{m-1} \\ = \; & a^{d+\delta}_{2,1}(\phi, d\theta^2) \cdot a^{d+\delta}_{m-1,m-1,0}(0, g_{D^{m-1}}) \\ = \; & \frac{1}{\sqrt{\pi}\mathrm{vol}(S^{m-2})(m-2)!}\phi_{;11}\mathfrak{L}_{m-1}\,. \end{aligned}$$

We solve for $c^{1,k}_{m+1,m,0}$ and $c^{2,k}_{m+1,m,0}$ to complete the proof. □

Proof of Lemma 3.8.6 (1): Let $m \ge 2$. We will use the method of universal examples to show that only the monomial $\mathrm{Tr}\,\{S^{m-1}\}$ is relevant in computing $\{a^{d+\delta}_{m,m,0}(0, g), a^{d+\delta}_{m+1,m,1}(0, g)\}$. To have a uniform notation, let

$$\begin{aligned} & P_{m-1}(g) := a^{d+\delta}_{m,m,0}(0, g), \quad c_{m-1} := c^0_{m,m,0}, \quad \mathfrak{c}_{m-1} := \mathfrak{c}^0_{m,m,0} \quad \text{or} \\ & P_{m-1}(g) := a^{d+\delta}_{m+1,m,1}(0, g),\; c_{m-1} := c^0_{m+1,m,1},\; \mathfrak{c}_{m-1} := \mathfrak{c}^0_{m+1,m,1}\,. \end{aligned}$$

Let $(y_1, ..., y_{m-1})$ be the usual coordinates on $\mathbb{R}^{m-1}$. Let $f(y)$ be a smooth even function function of y and let

$$M := \{(y, r) \in \mathbb{R}^m : r \ge f(y)\}\,.$$

Let $\{A_1, ..., A_{m-1}\}$ be distinct real constants. We choose f so that

$$f(0) = 0, \quad (\partial^y_i f)(0) = 0, \quad \text{and} \quad L_{ij}(0) = A_i\delta_{ij}\,. \tag{3.8.g}$$

Then

$$P_{m-1}(g)(0) = (m-1)!c_{m-1}\mathcal{A} \quad \text{where} \quad \mathcal{A} := A_1 ... A_{m-1}\,. \tag{3.8.h}$$

We have $R = 0$, $E = 0$ and $\Omega = 0$. Thus there exists a polynomial Q_m of total weight $m-1$ in the tangential covariant derivatives of $\{\chi, L, S\}$ so that

$$P_{m-1} = \mathfrak{Tr}\{Q_m(\cdot)\}\,.$$

Let $\tilde\nabla$ be the Levi-Civita connection of the boundary and let $\tilde R$ be the associated curvature. Let $\{e_1, ..., e_{m-1}\}$ be an orthonormal frame for $T\partial M$ so $e_i(0) = \partial^y_i$. We must control $\tilde\nabla^k L$ for $k \ge 1$. Since the curvature of $\mathbb{R}^m$ vanishes, Lemma 1.1.4 shows that $\tilde\nabla L$ is a totally symmetric tensor field. The components of $\tilde\nabla^k L(0)$ are polynomials in the derivatives of the defining function f for any k. Let $\mathfrak{K}$ denote the ideal in the algebra of all polynomials in the jets of f which is generated by the monomials $\{A_1^2, ..., A_{m-1}^2\}$. In light of Equation (3.8.h), we shall work modulo $\mathfrak{K}$ since such elements do not contribute to $\mathcal{A}$.

We first study $\tilde\nabla^2 L$. This is not a symmetric tensor field. However,

$$\begin{aligned}&\tilde R_{b_1b_2b_3b_4} = L_{b_1b_4}L_{b_2b_3} - L_{b_1b_3}L_{b_2b_4}, \quad \text{and}\\ &L_{a_1a_2:a_3a_4} - L_{a_1a_2:a_4a_3} = \tilde R_{a_3a_4a_1a_5}L_{a_5a_2} + \tilde R_{a_3a_4a_2a_5}L_{a_5a_1}\,.\end{aligned}$$

This shows that $A^2_{a_5}$ divides $\{\tilde R_{a_3a_4a_1a_5}L_{a_5a_2} + \tilde R_{a_3a_4a_2a_5}L_{a_5a_1}\}(0)$. Consequently $\tilde\nabla^2 L(0)$ is totally symmetric modulo the ideal $\mathfrak{K}$. Thus we may choose the 4 jets of f to kill the symmetrization of $(\tilde\nabla^2 L)(0)$ and thereby ensure $(\tilde\nabla^2 L)(0) \in \mathfrak{K}$. Similarly, by choosing the derivatives of order $k+2$ of f at the origin appropriately, we may suppose that

$$(\tilde\nabla^k L)(0) \in \mathfrak{K} \quad \text{for} \quad k > 0\,.$$

We therefore suppress $\tilde\nabla^k L$ henceforth. Since $\chi_{:a} = 2L_{ab}(\mathfrak{e}_b\mathfrak{i}_m + \mathfrak{e}_m\mathfrak{i}_b)$, further covariant differentiation of χ only involves covariantly differentiating the endomorphism $\mathfrak{e}_b\mathfrak{i}_m + \mathfrak{e}_m\mathfrak{i}_b$. Thus inductively there exist suitably chosen endomorphisms $\mathcal{E}_\star$ of weight 0 so

$$\chi_{:a_1\ldots a_k} = L_{a_1b_1}L_{a_2b_2}\ldots L_{a_kb_k}\mathcal{E}_{b_1\ldots b_k}\,.$$

If a $\chi_{:a_1\ldots}$ term appears, it must be contracted with another index a_1; the above equation contains no $L_{a_1a_1}$ term. Thus this contraction involves a different variable which produces an A^2_a term; such terms can be ignored in light of Equation (3.8.h) since we are working modulo the ideal $\mathfrak{K}$. Similarly as

$$S = \begin{cases} -L_{ab}\mathfrak{e}_b\mathfrak{i}_a & \text{on} \quad \Lambda(\mathbb{R}^{m-1}) \\ 0 & \text{on} \quad \Lambda(\mathbb{R}^{m-1}) \wedge dr\,. \end{cases}$$

$\tilde\nabla^k S$ plays no role if $k \geq 1$. If an $L_{a_1b_1}$ term appears where a_1 is not to be contracted with b_1, then A must be divisible by A^2_a. If the term L_{aa} appears in a monomial Q, then we may factor $Q = L_{aa}Q_0$ and then apply Lemma 1.8.10 (1) to see the supertrace of Q_0 vanishes. Thus L does not appear as a variable. This shows that only the monomial S^{m-1} is relevant. Consequently

$$P_{m-1}(g)(0) = \mathfrak{c}_{m-1}\mathfrak{Tr}\{S^{m-1}\}(0)\,. \tag{3.8.i}$$

Decompose $\Lambda(\mathbb{R}^{m-1}) = \Lambda(\mathbb{R}\cdot dy^1) \otimes \ldots \otimes \Lambda(\mathbb{R}\cdot dy^{m-1})$. Then

$$\begin{aligned}&S = \sum_{1\leq a\leq m-1} \mathrm{Id} \otimes \ldots \otimes \mathrm{Id} \otimes S_a \otimes \mathrm{Id} \otimes \ldots \otimes \mathrm{Id} \quad \text{where}\\ &S_a = 0 \text{ on } \Lambda^0(\mathbb{R}\cdot dy^a) \text{ and } S_a = -A_a \text{ on } \Lambda^1(\mathbb{R}\cdot dy^a)\,.\end{aligned}$$

One then has that

$$\mathfrak{Tr}(S^{m-1}) = (m-1)!\mathfrak{Tr}(S_1)\ldots\mathfrak{Tr}(S_{m-1}) = (m-1)!\mathcal{A}\,. \tag{3.8.j}$$

Assertion (1) of Lemma 3.8.6 follows from Equations (3.8.h), (3.8.j), and (3.8.i). $\square$

Proof of Lemma 3.8.6 (2): Let $m \geq 4$. To simplify the notation, we set

$$P_m(g) := a^{d+\delta}_{m+1,m,0}(0,g), \quad c_m := c^{3,0}_{m+1,m,0}, \quad \text{and} \quad \mathfrak{c}_m := \mathfrak{c}^3_{m+1,m,0}\,.$$

Let $(u_1, u_2, y_1, ..., y_{m-3}, r)$ be coordinates on $\mathbb{R}^m$. Let $f(y)$ satisfy the normalizations of Equation (3.8.g). We set

$$\begin{aligned}
&M := \{x \in \mathbb{R}^m : r \geq f(y)\}, \quad \text{and} \\
&ds_M^2 := du^1 \circ du^1 + e^{-2f_2(u_1,r)} du^2 \circ du^2 \\
&\qquad\qquad + dy^1 \circ dy^1 + ... + dy^{m-3} \circ dy^{m-3} + dr \circ dr \,.
\end{aligned}$$

where f_2 is a suitably chosen cubic polynomial. We then have $R(\cdot)(0) = 0$ and the non-vanishing components of L and ∇R at the origin are given, up to the usual $\mathbb{Z}_2$ symmetries, by:

$$\begin{aligned}
&L(\partial_i^y, \partial_j^y)(0) = A_i \delta_{ij}, \quad \text{and} \\
&R(\partial_1^u, \partial_2^u, \partial_2^u, \partial_1^u; \partial_r) = R(\partial_1^u, \partial_2^u, \partial_2^u, \partial_r; \partial_1^u) = A_0 \,.
\end{aligned}$$

Consequently

$$P_m(g)(0) = 2(m-3)! c_m \mathcal{A} \text{ where } \mathcal{A} := A_0 A_1 ... A_{m-3} \,. \tag{3.8.k}$$

Let $\mathfrak{K}$ be the ideal generated by the elements $\{A_0^2, A_1^2, ..., A_{m-3}^2\}$. If we set $A_0 = 0$, then this manifold is a product of the manifold considered previously with a flat factor. This shows that $\nabla^k R(0)$, $\nabla^k E(0)$, $\nabla^k \Omega(0)$ are all divisible by A_0 for $k \geq 1$ and vanish if $k = 0$.

We consider terms which can give rise to $\mathcal{A}$ after taking the supertrace. Let $\mathcal{E}$ denote a generic polynomial in the tangential covariant derivatives of L, of S, and of χ when A_0 is set to zero. Since we are not interested in terms which are divisible by A_0^2 and since A_0 has weight 3, we factor out a term which can be linear in A_0 to express P_m symbolically in the form

$$\begin{aligned}
P_m \;=\; & \sum_{k\geq 1} \mathfrak{Tr}\{\nabla^k R \cdot \mathcal{E}_{m-k-2}^R\} + \sum_{k\geq 1} \mathfrak{Tr}\{\nabla^k E \cdot \mathcal{E}_{m-k-2}^E\} \\
+ & \sum_{k\geq 1} \mathfrak{Tr}\{\nabla^k \Omega \cdot \mathcal{E}_{m-k-2}^\Omega\} + \sum_{k\geq 2} \mathfrak{Tr}\{\tilde{\nabla}^k L \cdot \mathcal{E}_{m-k-1}^L\} \\
+ & \sum_{k\geq 2} \mathfrak{Tr}\{\tilde{\nabla}^k S \cdot \mathcal{E}_{m-k-1}^S\} + \sum_{k\geq 3} \mathfrak{Tr}\{\tilde{\nabla}^k \chi \cdot \mathcal{E}_{m-k}^\chi\} + \dots \,.
\end{aligned}$$

We set $A_0 = 0$ in studying the "coefficient" monomials $\mathcal{E}$. Thus the arguments given above in the proof of Assertions (1) of Lemma 3.8.6 show that only powers of S are relevant. This shows that we may express

$$\begin{aligned}
P_m = \mathfrak{Tr}\Big\{ & \sum_{k\geq 1} c_{k,R} \nabla^k R \cdot S^{m-k-2} + \sum_{k\geq 1} c_{k,E} \nabla^k E \cdot S^{m-k-2} \\
+ & \sum_{k\geq 1} c_{k,\Omega} \nabla^k \Omega \cdot S^{m-k-2} + \sum_{k\geq 2} c_{k,L} \tilde{\nabla}^k L \cdot S^{m-k-1} \\
+ & \sum_{k\geq 2} c_{k,S} \tilde{\nabla}^k S \cdot S^{m-k-1} + \sum_{k\geq 3} c_{k,\chi} \tilde{\nabla}^k \chi \cdot S^{m-k} \Big\} + \dots \,.
\end{aligned} \tag{3.8.l}$$

Furthermore, all auxiliary indices must be fully contracted within the variables

$$\{\nabla^k R,\ \nabla^k E,\ \nabla^k\Omega,\ \tilde\nabla^k L,\ \tilde\nabla^k S,\ \tilde\nabla^k\chi\}\,.$$

By Lemma 1.8.10 (1),

$$\mathfrak{Tr}\{S^k\}=0 \text{ for } k<m-1 \quad\text{and}\quad \mathfrak{Tr}\{ES^k\}=0 \text{ for } k<m-3\,. \tag{3.8.m}$$

Thus the terms in $\nabla^k R$ and $\tilde\nabla^k L$ do not appear in Equation (3.8.l) since, being scalars, they could be moved outside the supertrace. Since Ω is skew-adjoint while S is self-adjoint, this term does not appear. Terms involving $\tilde\nabla^k S$ must be fully contracted. Thus, modulo lower order terms which can be absorbed at an earlier stage, these terms in $\tilde\nabla^k S$ have the form

$$\begin{aligned} &\mathfrak{Tr}\{S_{:a_1a_1a_2a_2\dots}S^k\}=\tfrac{1}{k+1}\mathfrak{Tr}\{S^{k+1}\}_{:a_1a_1a_2a_2\dots} \bmod \mathfrak{K}\\ =\ &0 \bmod \mathfrak{K}\,.\end{aligned}$$

Similarly, we can eliminate the terms $\chi_{:a_1a_1a_2a_2\dots}S^k$ from Equation (3.8.l).

Extend S to be covariant constant along the geodesic normal rays from the boundary. This permits us to move covariant derivatives outside the trace once again and apply Equation (3.8.m) to conclude that the only relevant invariant is $E_{;m}S^{m-3}$. Consequently

$$P_m(g)(0)=\mathfrak{c}_m\mathfrak{Tr}\{E_{;m}S^{m-3}\}(0)\,. \tag{3.8.n}$$

We may decompose $\Lambda(\mathbb{R}^{m-1})=\Lambda(\mathbb{R}^2)\otimes\Lambda(\mathbb{R}^{m-3})$ and $E_{;m}=\tilde E_{;m}\otimes\tilde S$ for

$$E_{;m}(0)=\left\{\begin{array}{l} 0 \text{ on } \Lambda^0(\mathbb{R}^2)\oplus\Lambda^2(\mathbb{R}^2),\\ -A_0\mathrm{Id} \ \text{ on } \Lambda^1(\mathbb{R}^2)\,.\end{array}\right.$$

Consequently

$$\mathfrak{Tr}\{E_{;m}S^{m-3}\}(0)=2(m-3)!\mathcal{A}\,. \tag{3.8.o}$$

Lemma 3.8.6 (4) now follows from Equations (3.8.k), (3.8.n) and (3.8.o). □

Proof of Lemma 3.8.7: To prove Assertion (1), we use product formulae. Let $M_1=\mathbb{T}^{m-1}$ be the torus and let D_1 be the scalar Laplacian. Since the structures are flat,

$$a_{n,m-1}(x_1,D_1)=\left\{\begin{array}{rl} (4\pi)^{-(m-1)/2} & \text{if} \quad n=0,\\ 0 & \text{if} \quad n>0\,.\end{array}\right.$$

Let $(M_2,D_2):=([0,1],-\partial_r^2)$. Let $M:=M_1\times M_2$ and $D:=D_1+D_2$. Let $\mathcal{B}:=\nabla_{e_m}+S$ where S is constant and where e_m is the inward unit normal; $e_m=\partial_r$ when $r=0$ and $e_m=-\partial_r$ when $r=1$. Assertion (1) follows from the identity:

$$\begin{aligned} a_{n,m,k}(y,D,\mathcal{B}) &= \sum_{n_1+n_2=n} a_{n_1,m-1}(x_1,D_1)\cdot a_{n_2,1,k}(y_2,D_2,\mathcal{B})\\ &= (4\pi)^{-(m-1)/2}a_{n,1,k}(y_2,D_2,\mathcal{B})\,.\end{aligned}$$

In view of Assertion (1), it suffices to take $m = 1$ in the proof of the remaining assertions. Let

$$M := [0, 1], \quad D_0 := -\partial_r^2, \quad \text{and} \quad \mathcal{B} := \nabla_{e_m} + S_0 .$$

We choose f so that f vanishes identically near $r = 1$ so only the component $r = 0$ where ∂_r is the inward unit normal is relevant.

To prove Assertion (2), we consider a conformal variation $D_\varepsilon := e^{-2\varepsilon f} D_0$. Lemmas 3.1.14 and 3.5.4 imply that

$$\partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B})|_{\varepsilon=0} = (1-n) a_n(f, D_0, \mathcal{B}) \quad \text{and} \quad \partial_\varepsilon S_\varepsilon|_{\varepsilon=0} = -\tfrac{1}{2} f_{;m} .$$

For $n \geq 3$, $f_{;m} S^{n-2}$ arises only from $\partial_\varepsilon S_\varepsilon^{n-1}$ and not from any interior variation. We compute:

$$\begin{aligned}
&\partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) = \partial_\varepsilon \int_{\partial M} \mathfrak{c}_{n,1,0}^0 S^{n-1} dy|_{\varepsilon=0} + \dots \\
= \quad & -\tfrac{1}{2}(n-1)\mathfrak{c}_{n,1,0} \int_{\partial M} f_{;m} S^{n-2} dy + \dots \\
= \quad & (1-n) a_n(f, D_0, \mathcal{B}) = (1-n) c_{n,1,1}^0 \int_{\partial M} f_{;m} S^{n-2} dy + \dots .
\end{aligned}$$

This shows $\mathfrak{c}_{n,1,1}^0 = \frac{1}{2}\mathfrak{c}_{n,1,0}^0$ and establishes Assertion (2) of Lemma 3.8.7.

To prove Assertion (3), we consider a scalar variation $D_\varrho := D_0 - \varrho f$. By Lemma 3.1.15,

$$\partial_\varrho a_n(1, D_\varrho, \mathcal{B})|_{\varrho=0} = a_{n-2}(f, D_0, \mathcal{B}) .$$

If $n \geq 5$, then $f_{;m} S^{n-4}$ arises only from $\partial_\varepsilon E_{;m} S^{n-4}$. We compute:

$$\begin{aligned}
&\partial_\varrho a_n(1, D_\varrho, \mathcal{B})|_{\varrho=0} = \partial_\varepsilon \int_{\partial M} \mathfrak{c}_{n,1,0}^3 E_{;m} S^{n-4} dy|_{\varrho=0} + \dots \\
= \quad & \int_{\partial M} \mathfrak{c}_{n,1,0}^3 f_{;m} S^{n-4} dy + \dots \\
= \quad & a_{n-2}(f, D_0, \mathcal{B}) = \int_{\partial M} \mathfrak{c}_{n-2,1,1}^0 f_{;m} S^{n-4} dy + \dots .
\end{aligned}$$

This shows $\mathfrak{c}_{n,1,0}^3 = \mathfrak{c}_{n-2,1,1}^0$. □

The remainder of this section is devoted to the proof of Theorem 3.8.3. Assertions (1), (3), and (4) are now immediate from Lemma 3.8.5. We shall apply this Lemma together with Lemmas 3.8.6 and 3.8.7 to prove the remaining assertions. By Lemma 3.8.5, we set k=0.

Computation of $c_{m+1,m,0}^3$. Suppose first $m = 3$. Then one has that

$$\begin{aligned}
\mathcal{F}_{3,3}^{3,0} \quad &= \quad 2R_{1223;1} + 2R_{2113;2} = 2R_{1223;1} - 2R_{1213;2} \\
&= \quad -2R_{1212;3} = R_{abba;m} .
\end{aligned}$$

Thus only the term $R_{abba;m}$ is relevant. We have

$$\operatorname{Tr}_{\Lambda^1}(240\Pi_+ - 120\Pi_-)E_{;m} = -240 R_{abba;m} + \dots,$$

$$\begin{aligned}
&\mathrm{Tr}_{\Lambda^2}(240\Pi_+ - 120\Pi_-)E_{;m} = 120R_{abba;m} + ...,\\
&a^{d+\delta}_{4,3,0}(0,g) = \int_{\partial M}(4\pi)^{-3/2}\tfrac{1}{360}(240+120)R_{abba;m}dy + ...\\
&c^{3,0}_{4,3,0} = \tfrac{1}{2\sqrt{\pi}}\tfrac{1}{4\pi} = \tfrac{1}{4\sqrt{\pi}\mathrm{vol}\,(S^1)1!}\,.
\end{aligned}$$

On the other hand, if $m > 3$, then

$$\begin{aligned}
c^{3,0}_{m+1,m,0} &= \mathfrak{c}^{3}_{m+1,m,0} = \mathfrak{c}^{0}_{m-1,m,0} = \tfrac{1}{2}\mathfrak{c}^{0}_{m-1,m,0}\\
&= \tfrac{1}{2}\tfrac{1}{2\sqrt{\pi}}c^{0}_{m-1,m-1,0} = \tfrac{1}{4\sqrt{\pi}\mathrm{vol}\,(S^{m-2})(m-2)!}\,. \qquad \square
\end{aligned}$$

Computation of $c^0_{m+1,m,1}$. First suppose $m = 1$. Applying Theorem 3.6.1 to $[0,1]$ yields

$$a_2(f,\Delta^0,\mathcal{B}_a) - a_2(f,\Delta^1,\mathcal{B}_a) = \tfrac{1}{12\sqrt{\pi}}\int_{\partial M}\Big\{(3f_{;m}) - (-3f_{;m})\Big\}dy$$

$$= \tfrac{1}{2\sqrt{\pi}}\int f_{;m}dy\,,$$

Therefore

$$c^0_{2,1,1} = \tfrac{\sqrt{\pi}}{\mathrm{vol}\,S^1}\,.$$

If $m > 1$, then Lemmas 3.8.5, 3.8.6, and 3.8.7 imply

$$\begin{aligned}
c^{0}_{m+1,m,1} &= \mathfrak{c}^{0}_{m+1,m,1} = \mathfrak{c}^{0}_{m+1,m,0} = \tfrac{2\sqrt{\pi}}{2}\mathfrak{c}^{0}_{m+1,m+1,0}\\
&= \sqrt{\pi}c^{0}_{m+1,m+1,0} = \tfrac{\sqrt{\pi}}{\mathrm{vol}\,(S^m)m!}\,. \quad \square
\end{aligned}$$

3.9 Leading terms in the asymptotics

In previous sections, we have determined exact formulae for certain of the heat trace coefficients. In this section, we present partial information for all the coefficients a_n with n sufficiently large. We begin our discussion with two theorems computing what has been called the *large energy limit*. These are the terms with a maximal number of covariant derivatives. These terms have also been called the *leading terms* in the asymptotics of the Laplacian; they play an important role in compactness theorems for isospectral deformations proved by various authors [106, 117, 118, 119, 294].

There are several proofs of the following two results. Gilkey [182] gave a combinatorial calculation, Branson et. al. [88] used functorial methods, and Avramidi used a modified de Witt Ansatz [12]. It is worth noting that the case $m = 2$ is due to Osgood et. al. [294]. We omit lower order terms involving fewer jets of the symbol. If $\mathcal{E}$ is a tensor, we use the *reduced or Bochner Laplacian* to define

$$\Delta^\nu\mathcal{E} := (-1)^\nu\mathcal{E}_{;i_1i_1i_2i_2...i_\nu i_\nu}\,.$$

Theorem 3.9.1 *Let D be an operator Laplace type. If $n \geq 1$, then modulo lower order terms,*

$$a_{2n}(x,D) = (4\pi)^{-m/2}\tfrac{(-1)^n}{2^{n+1}1\cdot 3\cdot\ldots\cdot(2n+1)}\Delta^{n-1}\mathrm{Tr}\,\{-(8n+4)E - 2n\tau\mathrm{Id}\,\} + \ldots$$

For example, we have

$$\begin{aligned}
a_2(x,D) &= \tfrac{1}{4\cdot 3}(4\pi)^{-m/2}\mathrm{Tr}\,\{12E + 2\tau\mathrm{Id}\,\} + \ldots, \\
a_4(x,D) &= \tfrac{1}{8\cdot 3\cdot 5}(4\pi)^{-m/2}\mathrm{Tr}\,\{20E_{;kk} + 4\tau_{;kk}\mathrm{Id}\,\} + \ldots, \\
a_6(x,D) &= \tfrac{1}{16\cdot 3\cdot 5\cdot 7}(4\pi)^{-m/2}\mathrm{Tr}\,\{28E_{;iijj} + 6\tau_{;iijj}\} + \ldots
\end{aligned}$$

which agrees with the formulae of Theorem 3.3.1.

Information is also available about the corresponding integrated invariants. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are tensors, we define

$$\nabla^k\mathcal{E}_1 \cdot \nabla^k\mathcal{E}_2 := \mathcal{E}_{1;i_1\ldots i_k}\mathcal{E}_{2;i_1\ldots i_k}\,.$$

Theorem 3.9.2 *Let D be an operator of Laplace type on a closed Riemannian manifold. If $n \geq 3$, then modulo lower order terms,*

$$\begin{aligned}
a_{2n}(1,D) \;&=\; (4\pi)^{-m/2}\tfrac{(-1)^n}{2^{n+1}1\cdot 3\cdot\ldots\cdot(2n+1)}\int_M \mathrm{Tr}\,\{(n^2-n-1)|\nabla^{n-2}\tau|^2\mathrm{Id} \\
&+\; 2|\nabla^{n-2}\rho|^2\mathrm{Id} + 4(2n+1)(n-1)\nabla^{n-2}\tau\cdot\nabla^{n-2}E \\
&+\; 2(2n+1)\nabla^{n-2}\Omega\cdot\nabla^{n-2}\Omega \\
&+\; 4(2n+1)(2n-1)\nabla^{n-2}E\cdot\nabla^{n-2}E + \ldots\}dx\,.
\end{aligned}$$

If the boundary of M is non-empty, we study the low energy limit to obtain information about the boundary terms involving the least number of covariant derivatives. We adopt the notation established in Section 3.8. The following result is due to M. Bordag et al. [75] who gave a different derivation of it:

Theorem 3.9.3 *Let $\mathcal{B}$ define Robin boundary conditions for an operator of Laplace type. Expand*

$$\begin{aligned}
a_{n,m,0}(y,D,\mathcal{B}) &= \mathfrak{c}^0_{n,m,0}\mathrm{Tr}\,\{S^{n-1}\} + \ldots \quad \textit{for} \quad n \geq 1 \\
a_{n,m,1}(y,D,\mathcal{B}) &= \mathfrak{c}^0_{n,m,1}\mathrm{Tr}\,\{S^{n-2}\} + \ldots \quad \textit{for} \quad n \geq 2\,.
\end{aligned}$$

1. *If $n = 1$, then $\mathfrak{c}^0_{n,m,0} = \frac{1}{4}(4\pi)^{(m-1)/2}$.*
2. *If $n \geq 2$, then $\mathfrak{c}^0_{n,m,0} = (4\pi)^{(n-m)/2}\frac{1}{\mathrm{vol}\,(S^{n-1})(n-1)!}$.*
3. *If $n = 2$, then $\mathfrak{c}^0_{n,m,1} = \frac{1}{2}(4\pi)^{-(m-1)/2}$.*
4. *If $n \geq 3$, then $\mathfrak{c}^0_{n,m,1} = \frac{1}{2}(4\pi)^{(n-m)/2}\frac{1}{\mathrm{vol}\,(S^{n-1})(n-1)!}$.*

Remark 3.9.4 By Lemma 1.1.8, we have that

$$\mathrm{vol}\,(S^1) = 2\pi, \quad \mathrm{vol}\,(S^2) = 4\pi, \quad \mathrm{vol}\,(S^3) = 2\pi^2, \quad \mathrm{vol}\,(S^4) = \tfrac{8}{3}\pi^2\,.$$

Thus we may use Theorem 3.9.3 to compute that

$$a^{d+\delta}_{2,m,0} = (4\pi)^{(2-m)/2}\tfrac{1}{\mathrm{vol}\,(S^1)1!}\mathrm{Tr}\,(S) + \ldots \;= 2(4\pi)^{-m/2}\mathrm{Tr}\,(S) + \ldots,$$

$$
\begin{aligned}
a^{d+\delta}_{3,m,0} &= (4\pi)^{(3-m)/2}\tfrac{1}{\mathrm{vol}\,(S^2)2!}\mathrm{Tr}\,(S^2) + ... = \tfrac{1}{2}(4\pi)^{-(m-1)/2}\mathrm{Tr}\,(S^2) + ...,\\
a^{d+\delta}_{4,m,0} &= (4\pi)^{(4-m)/2}\tfrac{1}{\mathrm{vol}\,(S^3)3!}\mathrm{Tr}\,(S^3) + ... = \tfrac{4}{3}(4\pi)^{-m/2}\mathrm{Tr}\,(S^3) + ...,\\
a^{d+\delta}_{5,m,0} &= (4\pi)^{(5-m)/2}\tfrac{1}{\mathrm{vol}\,(S^4)4!}\mathrm{Tr}\,(S^4) + ... = \tfrac{1}{4}(4\pi)^{-(m-1)/2}\mathrm{Tr}\,(S^4) + ...
\end{aligned}
$$

This agrees with the formulae given in Section 3.6 as does the calculation

$$
\begin{aligned}
a^{d+\delta}_{3,m,1} &= \tfrac{1}{4}(4\pi)^{-(m-1)/2}\mathrm{Tr}\,(S) + ... &&= \tfrac{1}{2}a^{d+\delta}_{3,m,0} + ...,\\
a^{d+\delta}_{4,m,1} &= \tfrac{2}{3}(4\pi)^{-m/2}\mathrm{Tr}\,(S^2) + ... &&= \tfrac{1}{2}a^{d+\delta}_{4,m,0} + ...,\\
a^{d+\delta}_{5,m,1} &= \tfrac{1}{8}(4\pi)^{-(m-1)/2}\mathrm{Tr}\,(S^3) + ... &&= \tfrac{1}{2}a^{d+\delta}_{5,m,0} + ...
\end{aligned}
$$

3.9.1 Proof of Theorem 3.9.1

Express

$$a_{2n}(x,D) = (4\pi)^{-m/2}\mathrm{Tr}\,\{c_{1,n}\Delta^{n-1}E + c_{2,n}\Delta^{n-1}\tau\mathrm{Id}\,\} + ...$$

where we have omitted terms that are quadratic or higher order in the structures $\{E, R, \Omega\}$ and that therefore involve at most $2n-4$ covariant derivatives.

We first determine $c_{1,n}$. We adopt the notation of Lemma 3.1.17. Let b be a smooth real valued function on S^1. Set

$$
\begin{aligned}
&A := \partial_\theta - b, && D_1 := A^*A = -(\partial_\theta^2 - \partial_\theta b - b^2), && E_1 := -\partial_\theta b - b^2,\\
&A^* := -\partial_\theta - b, && D_2 := AA^* = -(\partial_\theta^2 + \partial_\theta b - b^2), && E_2 := \ \ \partial_\theta b - b^2\,.
\end{aligned}
$$

We compute that

$$
\begin{aligned}
& (2n-1)\{a_{2n}(\theta,D_1) - a_{2n}(\theta,D_2)\}\\
= \ & (2n-1)(4\pi)^{-1/2}\{c_{1,n}\Delta^{n-1}E_1 - c_{1,n}\Delta^{n-1}E_2 + ...\}\\
= \ & (2n-1)(4\pi)^{-1/2}\{-2c_{1,n}\Delta^{n-1}\partial_\theta b + ...\}\,.
\end{aligned}
$$

We use the recursion relation given in Lemma 3.1.17 to equate this to

$$
\begin{aligned}
& (\partial_\theta^2 - 2\partial_\theta b)a_{2n-2}(x,D_1) = (4\pi)^{-1/2}\partial_\theta^2\{-c_{1,n-1}\Delta^{n-2}\partial_\theta b + ...\}\\
= \ & (4\pi)^{-1/2}\{c_{1,n-1}\Delta^{n-1}\partial_\theta b + ...\}\,.
\end{aligned}
$$

This implies that

$$-2(2n-1)c_{1,n} = c_{1,n-1}\,. \tag{3.9.a}$$

By Theorem 3.3.1, $c_{1,1} = 1$. We use induction and Equation (3.9.a) to see

$$c_{1,n} = \tfrac{(-1)^{n-1}}{2^{n-1}\cdot 1\cdot 3\cdot\ldots\cdot(2n-1)} = -(8n+4)\tfrac{(-1)^n}{2^{n+1}1\cdot 3\cdot\ldots\cdot(2n+1)}\,.$$

Next we study $c_{2,n}$. Let $m = 2n+2$, let $M := S^1 \times ... \times S^1$ be the flat product torus, and let Δ be the flat scalar Laplacian on M. For F and f smooth functions on M, define the variations

$$F_\varepsilon := e^{-2\varepsilon f}F \quad \text{and} \quad D_\varepsilon := e^{-2\varepsilon f}\Delta\,.$$

By Lemma 3.1.16

$$\partial_\varepsilon a_{2n}(F_\varepsilon, D_\varepsilon)|_{\varepsilon=0} = 0\,. \tag{3.9.b}$$

Since $\tau_0 = 0$ and $E_0 = 0$, we may use Lemma 3.3.2 to see that

$$\begin{aligned} &\partial_\varepsilon\{F_\varepsilon\Delta_\varepsilon^{n-1}E_\varepsilon\}|_{\varepsilon=0} = -\tfrac{1}{2}(m-2)F\Delta^n f + ..., &(3.9.c)\\ &\partial_\varepsilon\{F_\varepsilon\Delta_\varepsilon^{n-1}\tau_\varepsilon\}|_{\varepsilon=0} = 2(m-1)F\Delta^n f + ... \end{aligned}$$

We use Equation (3.9.b) and Equation (3.9.c) to conclude that

$$0 = \left\{(-nc_{1,n} + 2(2n+1)c_{2,n})\int_M F\Delta^n f dx + ...\right\}.$$

Since F and f were arbitrary, the coefficient of $F\Delta^n f$ must vanish. This completes the proof of Theorem 3.9.1 by showing

$$c_{2,n} = \tfrac{n}{2(2n+1)}c_{1,n} = -2n\tfrac{(-1)^n}{2^{n+1}1\cdot3\cdot...\cdot(2n+1)}\,. \qquad \square$$

3.9.2 Proof of Theorem 3.9.2

Let $n \geq 3$. We wish to study the integrated global invariant term $a_n(1,D)$. Terms that are linear in the covariant derivatives of $\{E,\Omega,R\}$ integrate to zero since M is closed. Thus only the quadratic terms count in the high energy limit. Commuting covariant derivatives introduces lower order terms which may be ignored.

Since M is a closed manifold, we may integrate by parts to reduce the number of invariants to be considered. For example, we can use the first and second Bianchi identities to see that

$$\int_M |\nabla^{n-2}R|^2 dx = \int_M \left\{4|\nabla^{n-2}\rho|^2 - |\nabla^{n-2}\tau|^2 + ...\right\}dx \quad \text{for} \quad n \geq 3\,.$$

We absorb terms with fewer than $2n-4$ covariant derivatives into the error term and omit lower order terms. We use invariance theory, the Bianchi identities, and integration by parts to express

$$\begin{aligned} a_n(1,D) \;&=\; \tfrac{(-1)^n}{2^{n+1}1\cdot3\cdot...\cdot(2n+1)}(4\pi)^{-m/2}\int_M \mathrm{Tr}\Big\{c_{1,n}|\nabla^{n-2}\tau|^2\mathrm{Id}\\ &+\; c_{2,n}|\nabla^{n-2}\rho|^2\mathrm{Id} + c_{3,n}\nabla^{n-2}\tau\cdot\nabla^{n-2}E \qquad (3.9.d)\\ &+\; c_{4,n}\nabla^{n-2}\Omega\cdot\nabla^{n-2}\Omega + c_{5,n}\nabla^{n-2}E\cdot\nabla^{n-2}E + ...\Big\}dx\,. \end{aligned}$$

Let D be a scalar operator of Laplace type on M. Set $D_\varepsilon := e^{-2\varepsilon f}D$. Let

$$\begin{aligned} \mathcal{E}_\tau &:= -\int_M \left\{\nabla^{n-2}f_{;kk}\cdot\nabla^{n-2}\tau\right\}dx = \int_M \left\{f\Delta^{n-1}\tau + ...\right\}dx,\\ \mathcal{E}_E &:= -\int_M \left\{\nabla^{n-2}f_{;kk}\cdot\nabla^{n-2}E\right\}dx = \int_M \left\{f\Delta^{n-1}E + ...\right\}dx\,. \end{aligned}$$

By Lemma 3.1.14 and Theorem 3.9.1,

$$\begin{aligned} &\partial_\varepsilon a_{2n}(1,D_\varepsilon)|_{\varepsilon=0} = (m-2n)a_{2n}(f,D) \qquad (3.9.e)\\ &= \tfrac{(-1)^n(m-2n)}{2^{n+1}1\cdot3\cdot...\cdot(2n+1)}(4\pi)^{-m/2}\int_M \Big\{-(8n+4)\mathcal{E}_E - 2n\mathcal{E}_\tau + \dots\Big\}dx\,. \end{aligned}$$

Note that $2\rho_{ij;j} = \tau_{;k}$. We use Lemma 3.3.2 to compute that

$$c_{1,n}\partial_\varepsilon \int_M (\nabla^{n-2}\tau_\varepsilon \cdot \nabla^{n-2}\tau_\varepsilon)dx|_{\varepsilon=0} = 4(m-1)c_{1,n}\mathcal{E}_\tau + \dots \quad (3.9.f)$$

$$c_{2,n}\partial_\varepsilon \int_M (\nabla^{n-2}\rho_\varepsilon \cdot \nabla^{n-2}\rho_\varepsilon)dx|_{\varepsilon=0} = mc_{2,n}\mathcal{E}_\tau + \dots$$

$$c_{3,n}\partial_\varepsilon \int_M (\nabla^{n-2}\tau_\varepsilon \cdot \nabla^{n-2}E_\varepsilon)dx|_{\varepsilon=0} \quad (3.9.g)$$

$$= c_{3,n}\{-\tfrac{m-2}{2}\mathcal{E}_\tau + 2(m-1)\mathcal{E}_E + \dots\}$$

$$c_{4,n}\partial_\varepsilon \int_M (\nabla^{n-2}\Omega_\varepsilon \cdot \nabla^{n-2}\Omega_\varepsilon)dx|_{\varepsilon=0} = 0 + \dots$$

$$c_{5,n}\partial_\varepsilon \int_M (\nabla^{n-2}E_\varepsilon \cdot \nabla^{n-2}E_\varepsilon)dx|_{\varepsilon=0} = -c_{5,n}(m-2)\mathcal{E}_E + \dots .$$

We use Equation (3.9.e) and Display (3.9.f). By equating coefficients of $\mathcal{E}_E$ and $\mathcal{E}_\tau$, the following relations are obtained

$$(8n+4)(2n-m) = 2(m-1)c_{3,n} - (m-2)c_{5,n} \quad \text{and}$$
$$2n(2n-m) = 4(m-1)c_{1,n} + mc_{2,n} - \tfrac{1}{2}(m-2)c_{3,n} .$$

We solve these equations to see

$$c_{1,n} = (n^2 - n - 1), \qquad c_{2,n} = 2,$$
$$c_{3,n} = 4(2n+1)(n-1), \quad c_{5,n} = 4(2n+1)(2n-1) .$$

Let $m = 2$. By Lemmas 1.2.5 and 1.2.6, E and Ω vanish on $\Lambda^0(M)$ and $\Lambda^2(M)$ while on $\Lambda^1(M)$ we have that

$$E = \begin{pmatrix} -R_{1221} & 0 \\ 0 & -R_{1221} \end{pmatrix} \quad \text{and} \quad \Omega_{12} = \begin{pmatrix} 0 & R_{1221} \\ -R_{1221} & 0 \end{pmatrix} .$$

Consequently since the scalar terms cancel in the supertrace we have by Theorem 1.3.9,

$$\begin{aligned} 0 &= a_{2n}(1,\Delta^0) - a_{2n}(1,\Delta^1) + a_{2n}(1,\Delta^0) \\ &= -\tfrac{(-1)^n}{2^{n+1}1\cdot 3\cdot \ldots \cdot (2n+1)}(4\pi)^{-1}\int_M \operatorname{Tr}_{\Lambda^1(M)}\Big\{c_{3,n}\nabla^{n-2}\tau\cdot\nabla^{n-2}E \\ &+ c_{4,n}\nabla^{n-2}\Omega\cdot\nabla^{n-2}\Omega + c_{5,n}\nabla^{n-2}E\cdot\nabla^{n-2}E + \ldots\Big\}dx \\ &= -\tfrac{(-1)^n}{2^{n+1}1\cdot 3\cdot \ldots \cdot (2n+1)}(-c_{3,n} - c_{4,n} + \tfrac{1}{2}c_{5,n}) \\ &\cdot (4\pi)^{-1}\int_M \Big\{\nabla^{n-2}\tau\cdot\nabla^{n-2}\tau\Big\}dx . \end{aligned}$$

We use this relation to complete the proof of Theorem 3.9.3 by computing

$$c_{4,n} = -c_{3,n} + \tfrac{1}{2}c_{5,n} = 2(2n+1) . \qquad \square$$

3.9.3 Proof of Theorem 3.9.3

We adopt the notation of Section 3.8.2 and expand, for $k = 0, 1$,

$$a_{n,m,k}(y, D, \mathcal{B}) = \mathfrak{c}^0_{n,m,0} \mathrm{Tr}\,(S^{n-k-1} + \ldots)\,.$$

We may use Theorem 3.5.1 to see that

$$\begin{array}{ll}
\mathfrak{c}^0_{1,m,0} = \frac{1}{4}(4\pi)^{(1-m)/2}, & \\
\mathfrak{c}^0_{2,m,0} = 2(4\pi)^{-m/2}, & \mathfrak{c}^0_{2,m,1} = \frac{1}{2}(4\pi)^{-m/2}, \\
\mathfrak{c}^0_{3,m,0} = \frac{1}{2}(4\pi)^{(1-m)/2}, & \mathfrak{c}^0_{3,m,1} = \frac{1}{4}(4\pi)^{(1-m)/2}, \\
\mathfrak{c}^0_{4,m,0} = \frac{4}{3}(4\pi)^{(1-m)/2}, & \mathfrak{c}^0_{4,m,1} = \frac{2}{3}(4\pi)^{(1-m)/2}\,.
\end{array}$$

We may therefore assume $n \geq 5$ in the proof of Theorem 3.9.3. We use Lemmas 3.8.5, 3.8.6 and 3.8.7 to compute

$$\begin{aligned}
\mathfrak{c}^0_{n,m,0} &= (4\pi)^{-(m-1)/2}\mathfrak{c}^0_{n,1,0} = (4\pi)^{-(m-n)/2}\mathfrak{c}^0_{n,n,0} \\
&= (4\pi)^{-(m-n)/2}c^0_{n,n,0} = (4\pi)^{-(m-n)/2}\tfrac{1}{\mathrm{vol}\,(S^{n-1})(n-1)!}, \\
\mathfrak{c}^0_{n,m,1} &= \tfrac{1}{2}\mathfrak{c}^0_{n,m,0} = (4\pi)^{-(m-n)/2}\tfrac{1}{2\mathrm{vol}\,(S^{n-1})(n-1)!}\,. \qquad \square
\end{aligned}$$

3.10 Heat trace asymptotics for transmission boundary conditions

We adopt the notation of Equation (3.2.a) and consider structures

$$(M, g, V, D, f) = ((M_+, g_+, V_+, D_+, f_+), (M_-, g_-, V_-, D_-, f_-))\,.$$

Here $g_\pm$ are Riemannian metrics on compact manifolds $M_\pm$, $V_\pm$ are smooth vector bundles over $M_\pm$, and $D_\pm$ are operators of Laplace type on $V_\pm$, respectively. The functions $f_\pm$ are assumed smooth over the manifolds $M_\pm$. We impose the compatibility conditions of Equations (3.2.b) and (3.2.c):

$$\begin{array}{c}
\partial M_+ = \partial M_- = \Sigma, \quad g_+|_\Sigma = g_-|_\Sigma, \\
V_+|_\Sigma = V_-|_\Sigma = V_\Sigma, \quad f_+|_\Sigma = f_-|_\Sigma\,.
\end{array}$$

No matching condition is assumed on the normal derivatives of f or of g on the interface Σ.

We use *transmission boundary conditions* as discussed in Section 1.6.1. Assume given an impedance matching endomorphism U defined on the hypersurface Σ. Set

$$\mathcal{B}_U\phi := \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \quad \oplus \quad \left\{\nabla_{\nu_+}\phi_+|_\Sigma + \nabla_{\nu_-}\phi_-|_\Sigma - U\phi_+|_\Sigma\right\}\,.$$

Let

$$\omega_a := \nabla^+_a - \nabla^-_a\,.$$

Since the difference of two connections is tensorial, ω_a is a well defined endomorphism of V_Σ. The tensor ω_a is *chiral*; it changes sign if the roles of + and

$-$ are reversed. On the other hand, the tensor field U is *non-chiral* as it is not sensitive to the roles of $+$ and $-$.

The following result is due to Gilkey, Kirsten, and Vassilevich [200]; see also related work by Bordag and Vassilevich [77] and Moss [286]. To simplify the notation, we define invariants reflecting the chirality

$$\begin{array}{ll}
\mathcal{L}_{ab}^{\text{even}} := L_{ab}^{+} + L_{ab}^{-}, & \mathcal{L}_{ab}^{\text{odd}} := L_{ab}^{+} - L_{ab}^{-}, \\
\mathcal{F}_{;\nu}^{\text{even}} := f_{;\nu^{+}} + f_{;\nu^{-}}, & \mathcal{F}_{;\nu}^{\text{odd}} := f_{;\nu^{+}} - f_{;\nu^{-}}, \\
\mathcal{F}_{;\nu\nu}^{\text{even}} := f_{;\nu^{+}\nu^{+}} + f_{;\nu^{-}\nu^{-}}, & \mathcal{F}_{;\nu\nu}^{\text{odd}} := f_{;\nu^{+}\nu^{+}} - f_{;\nu^{-}\nu^{-}}, \\
\mathcal{E}^{\text{even}} := E^{+} + E^{-}, & \mathcal{E}^{\text{odd}} := E^{+} - E^{-}, \\
\mathcal{E}_{;\nu}^{\text{even}} := E_{;\nu^{+}}^{+} + E_{;\nu^{-}}^{-}, & \mathcal{E}_{;\nu}^{\text{odd}} := E_{;\nu^{+}}^{+} - E_{;\nu^{-}}^{-}, \\
\mathcal{R}_{ijkl}^{\text{even}} := R_{ijkl}^{+} + R_{ijkl}^{-}, & \mathcal{R}_{ijkl}^{\text{odd}} := R_{ijkl}^{+} - R_{ijkl}^{-} \\
\Omega_{ij}^{\text{even}} := \Omega_{ij}^{+} + \Omega_{ij}^{-}, & \Omega_{ij}^{\text{odd}} := \Omega_{ij}^{+} - \Omega_{ij}^{-}.
\end{array}$$

Theorem 3.10.1 *Adopt the notation established above.*

1. $a_0(f,D,\mathcal{B}_U) = (4\pi)^{-m/2}\int_M f\operatorname{Tr}(\operatorname{Id})dx$.

2. $a_1(f,D,\mathcal{B}_U) = 0$.

3. $a_2(f,D,\mathcal{B}_U) = (4\pi)^{-m/2}\frac{1}{6}\int_M f\operatorname{Tr}\{\tau\operatorname{Id} + 6E\}dx$
$+(4\pi)^{-m/2}\frac{1}{6}\int_\Sigma 2f\operatorname{Tr}\{\mathcal{L}_{aa}^{\text{even}}\operatorname{Id} - 6U\}dy$.

4. $a_3(f,D,\mathcal{B}_U) = (4\pi)^{(1-m)/2}\frac{1}{384}\int_\Sigma \operatorname{Tr}\{f[\frac{3}{2}\mathcal{L}_{aa}^{\text{even}}\mathcal{L}_{bb}^{\text{even}} + 3\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{ab}^{\text{even}}]\operatorname{Id}$
$+9\mathcal{L}_{aa}^{\text{even}}\mathcal{F}_{;\nu}^{\text{even}}\operatorname{Id} + 48fU^2 + 24f\omega_a\omega_a - 24f\mathcal{L}_{aa}^{\text{even}}U - 24\mathcal{F}_{;\nu}^{\text{even}}U\}dy$.

5. $a_4(f,D,\mathcal{B}_U) = (4\pi)^{-m/2}\frac{1}{360}\int_M f\operatorname{Tr}\{60E_{;kk} + 60R_{ijji}E + 180E^2$
$+30\Omega_{ij}\Omega_{ij} + [12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2]\operatorname{Id}\}dx$
$+(4\pi)^{-m/2}\frac{1}{360}\int_\Sigma \operatorname{Tr}\{[-5\mathcal{R}_{ijji}^{\text{odd}}\mathcal{F}_{;\nu}^{\text{odd}} + 2\mathcal{R}_{a\nu a\nu}^{\text{odd}}\mathcal{F}_{;\nu}^{\text{odd}} - 5\mathcal{L}_{aa}^{\text{odd}}\mathcal{L}_{bb}^{\text{even}}\mathcal{F}_{;\nu}^{\text{odd}}$
$-\mathcal{L}_{ab}^{\text{odd}}\mathcal{L}_{ab}^{\text{even}}\mathcal{F}_{;\nu}^{\text{odd}} + \frac{18}{7}\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{ab}^{\text{even}}\mathcal{F}_{;\nu}^{\text{even}} - \frac{12}{7}\mathcal{L}_{aa}^{\text{even}}\mathcal{L}_{bb}^{\text{even}}\mathcal{F}_{;\nu}^{\text{even}}$
$+12\mathcal{L}_{aa}^{\text{even}}\mathcal{F}_{;\nu\nu}^{\text{even}}]\operatorname{Id} + f[-\mathcal{L}_{ab}^{\text{odd}}\mathcal{L}_{ab}^{\text{odd}}\mathcal{L}_{cc}^{\text{even}} - \mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{ab}^{\text{odd}}\mathcal{L}_{cc}^{\text{odd}}$
$+2\mathcal{L}_{ab}^{\text{odd}}\mathcal{L}_{bc}^{\text{odd}}\mathcal{L}_{ac}^{\text{even}} + 2\mathcal{R}_{abcb}^{\text{odd}}\mathcal{L}_{ac}^{\text{odd}} + 12\mathcal{R}_{ijji;\nu}^{\text{even}}$
$+\frac{40}{21}\mathcal{L}_{aa}^{\text{even}}\mathcal{L}_{bb}^{\text{even}}\mathcal{L}_{cc}^{\text{even}} - \frac{4}{7}\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{cc}^{\text{even}} + \frac{68}{21}\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{bc}^{\text{even}}\mathcal{L}_{ac}^{\text{even}}$
$+24\mathcal{L}_{aa:bb}^{\text{even}} + 10\mathcal{R}_{ijji}^{\text{even}}\mathcal{L}_{aa}^{\text{even}} + 2\mathcal{R}_{a\nu a\nu}^{\text{even}}\mathcal{L}_{aa}^{\text{even}} - 6\mathcal{R}_{a\nu b\nu}^{\text{even}}\mathcal{L}_{ab}^{\text{even}}$
$+2\mathcal{R}_{abcb}^{\text{even}}\mathcal{L}_{ac}^{\text{even}}]\operatorname{Id} + 18\omega_a^2\mathcal{F}_{;\nu}^{\text{even}} - 30\mathcal{E}^{\text{odd}}\mathcal{F}_{;\nu}^{\text{odd}} + 15U\mathcal{L}_{aa}^{\text{odd}}\mathcal{F}_{;\nu}^{\text{odd}}$
$-30U\mathcal{F}_{;\nu\nu}^{\text{even}} - 9U\mathcal{L}_{aa}^{\text{even}}\mathcal{F}_{;\nu}^{\text{even}} + 30U^2\mathcal{F}_{;\nu}^{\text{even}} + f[12\omega_a^2\mathcal{L}_{bb}^{\text{even}} + 24\omega_a\omega_b\mathcal{L}_{ab}^{\text{even}}$
$+60\mathcal{E}_{;\nu}^{\text{even}} - 60\omega_a\Omega_{a\nu}^{odd} + 60\mathcal{E}^{\text{even}}\mathcal{L}_{aa}^{\text{even}} - 60U^3 - 30U\mathcal{R}_{ijji}^{\text{even}} - 180U\mathcal{E}^{\text{even}}$
$-60U_{:aa} - 18U\mathcal{L}_{aa}^{\text{even}}\mathcal{L}_{bb}^{\text{even}} - 6U\mathcal{L}_{ab}^{\text{even}}\mathcal{L}_{ab}^{\text{even}} + 60U^2\mathcal{L}_{aa}^{\text{even}} - 60U\omega_a^2]\}dy$.

We shall omit the computation of a_4 in the interest of brevity and instead refer to [200] for details. We note that some information is available concerning a_5 in this setting.

3.10.1 Proof of Theorem 3.10.1

We may decompose

$$a_n(f,D,\mathcal{B}_U)=a_n^M(f,D)+a_n^\Sigma(f,D,\mathcal{B}_U)$$

where the interior integrals defining $a_n^M(f,D)$ are given by Theorem 3.3.1. We begin by determining the general form of the invariants a_n^Σ. There exist universal constants so that

$$\begin{aligned}
&a_0^\Sigma(f,D,U)=0,\\
&a_1^\Sigma(f,D,U)=\int_\Sigma c_1 f\mathrm{Tr}\Big\{\mathrm{Id}\Big\}dy\,,\\
&a_2^\Sigma(f,D,U)=(4\pi)^{-m/2}\tfrac{1}{6}\int_\Sigma \mathrm{Tr}\Big\{d_1 f\mathcal{L}_{aa}^{\mathrm{even}}\,\mathrm{Id}\,+d_2\mathcal{F}_{;\nu}^{\mathrm{even}}\,\mathrm{Id}\,+d_3 fU\Big\}dy\,,\\
&a_3^\Sigma(f,D,U)=(4\pi)^{(1-m)/2}\tfrac{1}{384}\int_\Sigma \mathrm{Tr}\Big\{fc_2\mathcal{L}_{aa}^{\mathrm{odd}}\,L_{bb}^{\mathrm{odd}}\,\mathrm{Id}\,+fc_3\mathcal{L}_{ab}^{\mathrm{odd}}\,L_{ab}^{\mathrm{odd}}\,\mathrm{Id}\\
&\quad+c_4\mathcal{L}_{aa}^{\mathrm{odd}}\,\mathcal{F}_{;\nu}^{\mathrm{odd}}\,\mathrm{Id}\,+c_5\mathcal{F}_{;\nu\nu}^{\mathrm{even}}\,\mathrm{Id}\,+c_6 f\mathcal{E}^{\mathrm{even}}\,+c_7 f\mathcal{R}_{ijji}^{\mathrm{even}}\,\mathrm{Id}\\
&\quad+c_8 f\mathcal{R}_{a\nu\nu a}^{\mathrm{even}}\mathrm{Id}\,+d_4 f\mathcal{L}_{aa}^{\mathrm{even}}\,\mathcal{L}_{bb}^{\mathrm{even}}\,\mathrm{Id}\,+d_5 f\mathcal{L}_{ab}^{\mathrm{even}}\,\mathcal{L}_{ab}^{\mathrm{even}}\,\mathrm{Id}\\
&\quad+d_6\mathcal{L}_{aa}^{\mathrm{even}}\,\mathcal{F}_{;\nu}^{\mathrm{even}}\,\mathrm{Id}\,+d_7 fU^2+d_8 f\mathcal{L}_{aa}^{\mathrm{even}}\,U+d_9\mathcal{F}_{;\nu}^{\mathrm{even}}\,U+e_1 f\omega_a\omega_a\Big\}dy\,.
\end{aligned}$$

We can simplify the task of determining these unknown coefficients at the very outset. We apply Lemma 3.2.1. Suppose D_0 is an operator of Laplace type on a closed Riemannian manifold M_0. Let Σ be a closed hypersurface of M_0 which separates M_0 into two components M_+ and M_-. Then we can regard $M=M_+\cup_\Sigma M_-$ and $D=D_+\cup_\Sigma D_-$. Set $U=0$. If f is smooth on M, then Σ plays no role in the computation of the heat trace asymptotics; the singularity is entirely “artificial”. This implies $a_n^\Sigma(f,D,\mathcal{B}_U)=0$. Consequently

$$c_1=c_2=c_3=c_4=c_5=c_6=c_7=c_8=0\,. \tag{3.10.a}$$

We can double the manifold to relate the heat trace invariants for the boundary condition $\mathcal{B}_U$ to the corresponding invariants for the decoupled Dirichlet and Robin boundary conditions. Let D_0 be an operator of Laplace type on a compact Riemannian manifold M_0 with boundary Σ. Let

$$\mathcal{B}_0\phi_0:=\phi_0|_{\partial M_0}\quad\text{and}\quad\mathcal{B}_S\phi_0:=(\nabla_{e_m}+S)\phi_0|_{\partial M_0}\,.$$

Let $M_\pm:=M_0$, $D_\pm:=D_0$, and let $U=-2S$. Let $M=M_+\cup_\Sigma M_-$. Extend $f_0\in C^\infty(M_0)$ as an even function f_{even} on M by setting

$$f_{\mathrm{even}}\,(x_\pm)=f_0(x)\,.$$

Then by Lemma 3.2.2,

$$a_n^\Sigma(f_{\mathrm{even}}\,,D,\mathcal{B}_U)=a_n^\Sigma(f_0,D_0,\mathcal{B}_D)+a_n^\Sigma(f_0,D_0,\mathcal{B}_{R(S)})\,.$$

The following observation is then an immediate consequence of Theorems 3.4.1 and 3.5.1.

$$
\begin{aligned}
&d_1 = 2, \quad && d_2 = 0, \quad && d_3 = -6, \\
&d_4 = \tfrac{3}{2}, \quad && d_5 = 3, \quad && d_6 = 9, \\
&d_7 = 48, \quad && d_8 = -24, \quad && d_9 = -24\,.
\end{aligned} \tag{3.10.b}
$$

To complete the proof of Theorem 3.10.1, we must determine e_1. We set $m = 2$ and use the index theorem for the de Rham complex. Set

$$\mathfrak{e}_i := \mathfrak{e}(e_i) \quad \text{and} \quad \mathfrak{i}_i = \mathfrak{i}(e_i)\,.$$

We consider $\Delta_\pm$ on $M_\pm$. By Lemma 1.6.3,

$$
\begin{aligned}
\omega_a &= (L_{ab}^+ + L_{ab}^-)\mathfrak{e}_m\mathfrak{i}_b + \mathfrak{i}_m\mathfrak{e}_b\,, \\
U &= (L_{ab}^+ + L_{ab}^-)\{\mathfrak{e}_m\mathfrak{i}_m\mathfrak{i}_a\mathfrak{e}_b + \mathfrak{i}_m\mathfrak{e}_m\mathfrak{e}_a\mathfrak{i}_b\}\,.
\end{aligned}
$$

We have relative to the basis $\{1, e_1, e_2, e_{12} := e_1 \wedge e_2\}$ for $\Lambda(M)|_\Sigma$ that

$$
\begin{aligned}
&\omega_1(1) = 0, \quad \omega_1(e_1) = \mathcal{L}_{aa}^{\text{even}}\, e_2, \quad \omega_1(e_2) = -\mathcal{L}_{aa}^{\text{even}}\, e_1, \quad \omega_1(e_{12}) = 0, \\
&U(1) = 0, \quad U(e_1) = \mathcal{L}_{aa}^{\text{even}}\, e_1, \quad U_1(e_2) = \mathcal{L}_{aa}^{\text{even}}\, e_2, \quad U(e_{12}) = 0\,.
\end{aligned}
$$

By Lemma 3.2.3,

$$0 = a_3(1, \Delta^0, \mathcal{B}_U) - a_3(1, \Delta^1, \mathcal{B}_U) + a_3(1, \Delta^2, \mathcal{B}_U)\,.$$

The scalar invariants cancel in the supertrace. Consequently

$$
\begin{aligned}
0 &= -\int_\Sigma \operatorname{Tr}_{\Lambda^1(M)}\left\{d_7U^2 + d_8\mathcal{L}_{aa}^{\text{even}}\, U + e_1\omega_1^2\right\}dy \\
&= (2d_7 + 2d_8 - 2e_1)\int_\Sigma \mathcal{L}_{aa}^{\text{even}}\, \mathcal{L}_{bb}^{\text{even}}\, dy\,.
\end{aligned}
$$

This shows that

$$96 - 48 - 2e_1 = 0 \quad \text{so} \quad e_1 = 24\,. \qquad \square$$

Remark 3.10.2 The coefficient e_1 was first computed by Moss [286] using very different methods.

3.11 Heat trace asymptotics for transfer boundary conditions

As in the previous section, we adopt the notation of Equation (3.2.a) and consider structures

$$(M, g, V, D) = ((M_+, g_+, V_+, D_+), (M_-, g_-, V_-, D_-))\,.$$

We assume only the compatibility conditions of Equation (3.2.b)

$$\partial M_+ = \partial M_- = \Sigma \quad \text{and} \quad g_+|_\Sigma = g_-|_\Sigma\,.$$

We do **not** assume an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$. Let $F_\pm$ be smooth endomorphisms of $V_\pm$; there is no assumed relation between F_+ and F_-. Let $\operatorname{Tr}_\pm$ denote the fiber trace on $V_\pm$.

We adopt the notation of Section 1.6.3 and impose transfer boundary conditions

$$\mathcal{B}_S\phi := \left\{\begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--} \end{pmatrix}\begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}\right\}\Big|_\Sigma .$$

We set $S_{+-} = S_{-+} = 0$ to introduce the associated decoupled Robin boundary conditions

$$\begin{aligned} \mathcal{B}_{R(S_{++})}\phi_+ &:= (\nabla^+_{\nu_+} + S_{++})\phi_+|_\Sigma, \quad \text{and} \\ \mathcal{B}_{R(S_{--})}\phi_- &:= (\nabla^-_{\nu_-} + S_{--})\phi_-|_\Sigma . \end{aligned}$$

Define the correction term $a_n(F, D, S)(y)$ by means of the identity

$$\begin{aligned} a_n(F, D, \mathcal{B}_S) &= \int_M a_n(F, D)(x)dx + \int_\Sigma a_n(F_+, D_+, \mathcal{B}_{R(S_{++})}dy \\ &+ \int_\Sigma a_n(F_-, D_-, \mathcal{B}_{R(S_{--})}dy + \int_\Sigma a_n(F, D, S)(y)dy . \end{aligned}$$

Since the interior invariants $a_n(F, D)$ are discussed in Theorem 3.3.1 and the invariants $a_n(F, D, \mathcal{B}_R)$ are discussed in Theorem 3.5.1, we must determine only the invariant $a_n(F, D, S)$ which measures the new interactions that arise from S_{+-} and S_{-+}. We follow the discussion of [201].

Theorem 3.11.1 *Adopt the notation established above.*

1. $a_n(F, D, S)(y) = 0$ *for* $n \le 2$.
2. $a_3(F, D, S)(y) = (4\pi)^{(1-m)/2}\frac{1}{2}\{\mathrm{Tr}_+(F_+S_{+-}S_{-+}) + \mathrm{Tr}_-(F_-S_{-+}S_{+-})\}$.
3. $a_4(F, D, S)(y) = (4\pi)^{-m/2}\frac{1}{360}\{$

$$\begin{aligned} &\mathrm{Tr}_+\{480(F_+S_{++} + S_{++}F_+)S_{+-}S_{-+} + 480F_+S_{+-}S_{--}S_{-+} \\ &\quad +(288F_+L^+_{aa} + 192F_+L^-_{aa} + 240F_{+;\nu_+})S_{+-}S_{-+}\} \\ &+\mathrm{Tr}_-\{480(F_-S_{--} + S_{--}F_-)S_{-+}S_{+-} + 480F_-S_{-+}S_{++}S_{+-} \\ &\quad +(288F_-L^-_{aa} + 192F_-L^+_{aa} + 240F_{-;\nu_-})S_{-+}S_{+-}\}\}. \end{aligned}$$

If $S_{+-} = 0$ and $S_{-+} = 0$, then $a_n(F, D, S) = 0$ as the boundary condition decouples and there is no interaction between M_+ and M_-. To ensure that the fiber trace is well defined, if S_{+-} appears, then S_{-+} must also appear. Since $a_n(F, D, S)$ is homogeneous of total weight $n-1$, we conclude $a_n(F, D, S) = 0$ for $n \le 2$. This establishes Assertion (1) of Theorem 3.11.1.

The heat trace coefficients must be symmetric with respect to interchanging the labels "+" and "−". We write down a basis of invariants of total weight 2 and 3 which involve both S_{+-} and S_{-+} to conclude

$$\begin{aligned} &a_3(F, D, \mathcal{B})(y) = (4\pi)^{-m/2}\tfrac{1}{384} \\ &\qquad c_0\{\mathrm{Tr}_+(F_+S_{+-}S_{-+}) + \mathrm{Tr}_-(F_-S_{-+}S_{+-})\}, \\ &a_4(F, D, \mathcal{B})(y) = (4\pi)^{-m/2}\tfrac{1}{360}\{ \\ &\qquad \tfrac{1}{2}c_{1,1}\{\mathrm{Tr}_+(F_+S_{++}S_{+-}S_{-+}) + \mathrm{Tr}_-(F_-S_{--}S_{-+}S_{+-})\} \end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}c_{1,2}\{\mathrm{Tr}_+(S_{++}F_+S_{+-}S_{-+})+\mathrm{Tr}_+(S_{--}F_-S_{-+}S_{+-})\}\\
&+c_2\{\mathrm{Tr}_+(F_+S_{+-}S_{--}S_{-+})+\mathrm{Tr}_-(F_-S_{-+}S_{++}S_{+-})\}\\
&+c_3\{L_{aa}^+\mathrm{Tr}_+(F_+S_{+-}S_{-+})+L_{aa}^-\mathrm{Tr}_-(F_-S_{-+}S_{+-})\}\\
&+c_4\{L_{aa}^-\mathrm{Tr}_+(F_+S_{+-}S_{-+})+L_{aa}^+\mathrm{Tr}_-(F_-S_{-+}S_{+-})\}\\
&+c_5\{\mathrm{Tr}_+(F_{+;\nu_+}S_{+-}S_{-+})+\mathrm{Tr}_-(F_{-;\nu_-}S_{-+}S_{+-})\}\}\,.
\end{aligned}
$$

We complete the proof of Theorem 3.11.1 by showing:

Lemma 3.11.2

1. *We have* $c_{1,1}=c_{1,2}$.
2. $c_0=192$.
3. $c_{1,1}=960$, $c_2=480$, $c_3+c_4=480$, *and* $c_5=240$.
4. $c_3=288$ *and* $c_4=192$.

Proof: We apply Lemma 3.1.4. If the structures in question are real, then $a_n(\cdot)$ is real. Thus all universal constants given above are real. Suppose the bundles $V_\pm$ are Hermitian, that the operators $D_\pm$ are formally self-adjoint, and that the endomorphisms $F_\pm$ are self-adjoint. If S_{++} and S_{--} are self-adjoint, and if S_{+-} is the adjoint of S_{-+}, then Lemma 1.6.4 implies $(D,\mathcal{B}_S)$ is self-adjoint and thus $a_n(\cdot)$ again is real; Assertion (1) now follows.

Since $c_{1,1}=c_{1,2}$, the lack of commutativity involved with dealing with endomorphisms plays no role; thus it suffices to consider the scalar case where everything is commutative. We assume therefore for the remainder of the proof that the bundles $V_\pm=M_\pm\times\mathbb{C}$ are trivial line bundles and that the operators $D_\pm$ are scalar. Thus we may drop "Tr" from the notation.

Let M_0 be a compact flat Riemannian manifold with smooth boundary and let Δ be the scalar Laplacian. We impose Robin boundary conditions defined by an auxiliary function S. By Theorem 3.5.1,

$$
\begin{aligned}
&a_3(f,\Delta,\mathcal{B}_{R(S)})(y)=(4\pi)^{(1-m)/2}\tfrac{1}{384}\mathrm{Tr}\,(192S^2+...),\\
&a_4(f,\Delta,\mathcal{B}_{R(S)})(y)=(4\pi)^{-m/2}\tfrac{1}{360}\mathrm{Tr}\,(480S^3\\
&\qquad+480S^2L_{aa}+240S^2f_{;m}+...)\,,
\end{aligned}
\tag{3.11.a}
$$

where we have suppressed terms which are not at least quadratic in S. We apply Lemma 3.2.4 to prove Assertion (2). Fix the parameter $\theta\in(0,\frac{\pi}{2})$. Let S_{++} and S_{+-} be arbitrary. Set

$$
\begin{aligned}
&S_{-+}:=S_{+-},\\
&S_{--}:=S_{++}+(\tan\theta-\cot\theta)\,S_{+-},\\
&S_\alpha:=S_{++}+\tan\theta\,S_{+-}=S_{--}+\cot\,\theta\,S_{-+},\\
&S_\beta:=S_{++}-\cot\,\theta S_{+-}=S_{--}-\tan\,\theta\,S_{-+}\,.
\end{aligned}
$$

Let $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ be Robin boundary conditions on M_0 defined by S_α and by S_β, respectively. Let $f_-=0$. Then

$$
a_n(f,\Delta,\mathcal{B}_S)=a_n(\cos^2\theta\,f_+,\Delta,\mathcal{B}_\alpha)+a_n(\sin^2\theta\,f_+,\Delta,\mathcal{B}_\beta)\,. \tag{3.11.b}
$$

We use the first identity of Display (3.11.a) to see

$$\begin{array}{rl} & a_3(\cos^2\theta f_+,\Delta,\mathcal{B}_\alpha)(y)+a_3(\sin^2\theta f_+,\Delta,\mathcal{B}_\beta)(y) \\ = & (4\pi)^{(1-m)/2}\frac{1}{384}\{f_+S_{+-}^2(192\cos^2\theta\tan^2\theta+192\sin^2\theta\cot^2\theta)+...\}(y) \\ = & (4\pi)^{(1-m)/2}\frac{1}{384}\{f_+(192S_{+-}^2)+...\}(y) \end{array}$$

where we have omitted terms in S_{++}^2 and $S_{++}S_{+-}$. On the other hand,

$$a_3(f_+,\Delta,\mathcal{B}_\mathcal{S})(y)=(4\pi)^{(1-m)/2}\frac{1}{384}c_0\{f_+S_{+-}^2\}(y)\,.$$

We use Equation (3.11.b) to equate these two expressions and complete the proof of Assertion (2) by showing

$$c_0=192\,.$$

The proof of Assertion (3) is similar. By Display (3.11.a),

$$\begin{array}{rl} & a_4(\cos^2\theta f_+,\Delta,\mathcal{B}_\alpha)(y)+a_4(\sin^2\theta f_+,\Delta,\mathcal{B}_\beta)(y) \\ = & \frac{1}{360}(4\pi)^{-m/2}\{480f_+(\cos^2\theta S_\alpha^3+\sin^2\theta S_\beta^3) \\ + & 480f_+L_{aa}(\cos^2\theta S_\alpha^2+\sin^2\theta S_\beta^2) \\ + & 240f_{+;\nu_+}(\cos^2\theta S_\alpha^2+\sin^2\theta S_\beta^2)\} \\ = & \frac{1}{360}(4\pi)^{-m/2}S_{+-}^2\{480f_+(3S_{++}+S_{+-}[\tan\theta-\cot\theta]) \\ + & 480f_+L_{aa}+240f^+_{;\nu_+}\}+\dots. \end{array}$$

On the other hand, we have by the defining relation that

$$\begin{array}{l} a_4(f_+,\Delta,\mathcal{B}_S)(y)=4\pi)^{-m/2}\frac{1}{360}S_{+-}^2\{c_{1,1}f_+S_{++} \\ \quad +c_2f_+(S_{++}+S_{+-}[\tan\theta-\cot\theta]) \\ \quad +(c_3+c_4)f_+L_{aa}+c_5f_{+;\nu_+}\}(y)+... \end{array}$$

Equation (3.11.b) permits one to equate these two expressions and thereby derive the following relations which establish Assertion (3)

$$c_{1,2}+c_2=3\cdot480,\quad c_2=480,\quad c_3+c_4=480,\quad\text{and}\quad c_5=240\,.$$

The missing information about $\{c_3,c_4\}$ is obtained via conformal transformations. We vary the structures on M_+ to define the one-parameter family

$$D(\varepsilon):=(e^{2\varepsilon f_+}D_+,D_-)\,.$$

To ensure that $g_+(\varepsilon)|_\Sigma=g_-|_\Sigma$, we assume $f_+|_\Sigma=0$. As the connections change, we must adjust $\mathcal{S}$ appropriately; in particular

$$S_{+-}(\varepsilon)=S_{+-}(0),\quad S_{-+}(\varepsilon)=S_{-+}(0),\quad\text{and}\quad S_{--}(\varepsilon)=S_{--}(0)\,.$$

We set $m=6$. By Lemma 3.1.16,

$$\partial_\varepsilon a_4(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D,\mathcal{B}_\mathcal{S})=0\,.$$

We apply Lemmas 3.4.3 and 3.5.4 and set $F=1$. One then has

$$\partial_\varepsilon|_{\varepsilon=0}S_{++}(\varepsilon)=\tfrac{m-2}{2}f_{+;\nu_+}=2f_{+;\nu_+},$$

$$\partial_\varepsilon|_{\varepsilon=0} L^+_{aa}(\varepsilon) = -(m-1) f_{+;\nu_+} = -5 f_{+;\nu_+},$$
$$\partial_\varepsilon|_{\varepsilon=0} \{\nabla_{\nu_+}(\varepsilon)(e^{-2\varepsilon f} F)\} = -2 f_{+;\nu_+}\,.$$

This then yields the relation $2c_{1,1} - 5c_3 - 2c_5 = 0$. Assertion (4) now follows from this relation and from Assertion (2). □

3.12 Time-dependent phenomena

Let $\mathfrak{D} = \{D_t\}$ be a time-dependent family of operators of Laplace type. We use the initial operator $D := D_0$ to define a reference metric g_0. Choose local frames $\{e_i\}$ for the tangent bundle of M and local frames $\{e_a\}$ for the tangent bundle of the boundary which are orthonormal with respect to the initial metric g_0. Use g_0 to define the measures dx on M and dy on ∂M. The metric g_0 defines the curvature tensor R and the second fundamental form L. We also use D to define a background connection ∇_0 that we use to multiply covariantly differentiate tensors of all types and to define the endomorphism E.

We assume given a decomposition of the boundary $\partial M = C_N \dot\sqcup C_D$ as the disjoint union of closed sets. It is permissible for either the Robin component C_N or the Dirichlet component C_D or both components to be empty. Motivated by the discussion in Section 2.9, we consider a 1 parameter family $\mathfrak{B} = \{\mathcal{B}_t\}$ of boundary operators which we expand formally in a Taylor series

$$\mathcal{B}_t\phi := \phi\Big|_{C_D} \oplus \left\{\phi_{;m} + S\phi + \sum_{r>0} t^r(\Gamma_{r,a}\phi_{;a} + S_r\phi)\right\}\Big|_{C_N}.$$

We consider the time-dependent heat equation described in Display (3.2.e) and let $K(t,x,\bar{x},\mathfrak{D},\mathfrak{B})$ be the associated kernel function. Let $a_n(f,\mathfrak{D},\mathfrak{B})$ be the heat trace asymptotics defined in Equation (3.2.f). In this section, we present results of [195] which extend Theorems 3.3.1, 3.4.1, and 3.5.1 to the time-dependent framework.

Expand $\mathfrak{D}$ in a Taylor series expansion

$$D_t u := Du + \sum_{r=1}^{\infty} t^r\left\{\mathcal{G}_{r,ij}u_{;ij} + \mathcal{F}_{r,i}u_{;i} + \mathcal{E}_r u\right\}.$$

By assumption, the operators $\mathcal{G}_{r,ij}$ are scalar. Our main result is the following theorem which describes the additional terms in the heat trace asymptotics which arise from the structures described by $\mathcal{G}_{r,ij}$, $\mathcal{F}_{r,i}$, $\mathcal{E}_r$, $\Gamma_{r,a}$, and S_r.

Theorem 3.12.1

1. $a_0(F,\mathfrak{D},\mathfrak{B}) = a_0(F,D,\mathcal{B})$ *and* $a_1(F,\mathfrak{D},\mathfrak{B}) = a_1(F,D,\mathcal{B})$.
2. $a_2(F,\mathfrak{D},\mathfrak{B}) = a_2(F,D,\mathcal{B}) + (4\pi)^{-m/2}\frac{1}{6}\int_M \operatorname{Tr}\{\frac{3}{2}F\mathcal{G}_{1,ii}\}dx.$

3. $a_3(F,\mathfrak{D},\mathfrak{B}) = a_3(F,D,\mathcal{B}) + (4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_D}\operatorname{Tr}\{-24F\mathcal{G}_{1,aa}\}dy$
$+(4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_N}\operatorname{Tr}\{24F\mathcal{G}_{1,aa}\}dy.$

4. $a_4(F,\mathfrak{D},\mathfrak{B}) = a_4(F,D,\mathcal{B}) + (4\pi)^{-m/2}\frac{1}{360}\int_M\operatorname{Tr}\{F(\frac{45}{4}\mathcal{G}_{1,ii}\mathcal{G}_{1,jj}$
$+\frac{45}{2}\mathcal{G}_{1,ij}\mathcal{G}_{1,ij} + 60\mathcal{G}_{2,ii} - 180\mathcal{E}_1 + 15\mathcal{G}_{1,ii}\tau - 30\mathcal{G}_{1,ij}\rho_{ij} + 90\mathcal{G}_{1,ii}E$
$+60\mathcal{F}_{1,i;i} + 15\mathcal{G}_{1,ii;jj} - 30\mathcal{G}_{1,ij;ij})\}dx + (4\pi)^{-m/2}\frac{1}{360}\int_{C_D}\operatorname{Tr}\{f(30\mathcal{G}_{1,aa}L_{bb}$
$-60\mathcal{G}_{1,mm}L_{bb} + 30\mathcal{G}_{1,ab}L_{ab} + 30\mathcal{G}_{1,mm;m} - 30\mathcal{G}_{1,aa;m} - 30\mathcal{F}_{1,m})$
$+F_{;m}(-45\mathcal{G}_{1,aa} + 45\mathcal{G}_{1,mm})\}dy + (4\pi)^{-m/2}\frac{1}{360}\int_{C_N}\operatorname{Tr}\{F(30\mathcal{G}_{1,aa}L_{bb}$
$+120\mathcal{G}_{1,mm}L_{bb} - 150\mathcal{G}_{1,ab}L_{ab} - 60\mathcal{G}_{1,mm;m} + 60\mathcal{G}_{1,aa;m}$
$+150\mathcal{F}_{1,m} + 180S\mathcal{G}_{1,aa} - 180S\mathcal{G}_{1,mm} + 360S_1)$
$+F_{;m}(45\mathcal{G}_{1,aa} - 45\mathcal{G}_{1,mm})\}dy.$

We use dimensional analysis to see that $\mathcal{G}_{r,ij}$ is homogeneous of weight $2r$, that $\mathcal{F}_{r,i}$ is homogeneous of weight $2r+1$, that $\mathcal{E}_r$ is homogeneous of weight $2r+2$, that $\Gamma_{r,a}$ is homogeneous of weight $2r$, and that S_r is homogeneous of weight $2r+1$. Thus these invariants do not enter into a_n for $n=0,1$. Assertion (1) of Theorem 3.12.1 now follows.

There exist universal constants so that

$$a_2(F,\mathfrak{D},\mathfrak{B}) = a_2(F,D,\mathcal{B}) + (4\pi)^{-m/2}\tfrac{1}{6}\int_M\operatorname{Tr}\left\{c_0F\mathcal{G}_{1,ii}\right\}dx, \qquad (3.12.a)$$

$$\begin{aligned}
&a_3(F,\mathfrak{D},\mathfrak{B}) = a_3(F,D,\mathcal{B}) + (4\pi)^{(1-m)/2}\tfrac{1}{384}\int_{C_D}\operatorname{Tr}\Big\{F(e_1^-\mathcal{G}_{1,aa}\\
&\qquad +e_2^-\mathcal{G}_{1,mm})\Big\}dy\\
&+(4\pi)^{(1-m)/2}\tfrac{1}{384}\int_{C_N}\operatorname{Tr}\Big\{F(e_1^+\mathcal{G}_{1,aa} + e_2^+\mathcal{G}_{1,mm})\Big\}dy,\\
&a_4(F,\mathfrak{D},\mathcal{B}) = a_4(F,D,\mathcal{B}) + (4\pi)^{-m/2}\tfrac{1}{360}\int_M\operatorname{Tr}\Big\{F(c_1\mathcal{G}_{1,ii}\mathcal{G}_{1,jj}\\
&\qquad +c_2\mathcal{G}_{1,ij}\mathcal{G}_{1,ij} + c_3\mathcal{G}_{2,ii} + c_4\mathcal{E}_1 + c_5\mathcal{G}_{1,ii}\tau + c_6\mathcal{G}_{1,ij}\rho_{ij}\\
&\qquad +c_7\mathcal{G}_{1,ii}E + c_8\mathcal{F}_{1,i;i} + c_9\mathcal{G}_{1,ii;jj} + c_{10}\mathcal{G}_{1,ij;ij})\Big\}dx\\
&+(4\pi)^{-m/2}\tfrac{1}{360}\int_{C_D}\operatorname{Tr}\Big\{F(e_3^-\mathcal{G}_{1,aa}L_{bb} + e_4^-\mathcal{G}_{1,mm}L_{bb}\\
&\qquad +e_5^-\mathcal{G}_{1,ab}L_{ab} + e_6^-\mathcal{G}_{1,mm;m} + e_7^-\mathcal{G}_{1,aa;m} + e_8^-\mathcal{G}_{1,am;a} + e_9^-\mathcal{F}_{1,m})\\
&\qquad +F_{;m}(e_{10}^-\mathcal{G}_{1,aa} + e_{11}^-\mathcal{G}_{1,mm})\Big\}dy\\
&+(4\pi)^{-m/2}\tfrac{1}{360}\operatorname{Tr}\Big\{F(e_3^+\mathcal{G}_{1,aa}L_{bb} + e_4^+\mathcal{G}_{1,mm}L_{bb} + e_5^+\mathcal{G}_{1,ab}L_{ab}\\
&\qquad +e_6^+\mathcal{G}_{1,mm;m} + e_7^+\mathcal{G}_{1,aa;m} + e_8^+\mathcal{G}_{1,am;a} + e_9^+\mathcal{F}_{1,m} + e_{12}^+S\mathcal{G}_{1,aa}
\end{aligned}$$

$$+e_{13}^+ S\mathcal{G}_{1,mm} + e_{14}^+ S_1 + e_{15}^+\Gamma_{1,a:a}) + F_{;m}(e_{10}^+\mathcal{G}_{1,aa} + e_{11}^+\mathcal{G}_{1,mm})\Big\}dy\,.$$

The product formulae of Lemma 3.2.5 show that the universal constants involved do not depend upon the dimension of the underlying manifold. We will complete the proof of Theorem 3.12.1 by evaluating these unknown coefficients. The possible non-commutativity in the vector valued case plays no role in these invariants. Consequently, for the remainder of this section, we shall assume that D is scalar valued, set $F = f$, and omit the "Tr" henceforth in the interest of notational simplicity.

We begin our study by using Lemma 3.2.6 and Theorem 3.6.1 to evaluate many of the coefficients:

Lemma 3.12.2 *1.* $c_0 = \frac{3}{2}$.

2. $e_1^- = -24$, $e_2^- = 0$, $e_1^+ = 24$, $e_2^+ = 0$.

3. $c_1 = \frac{45}{4}$, $c_2 = \frac{45}{2}$, $c_3 = 60$, $c_4 = -180$, $c_5 = 15$, $c_6 = -30$, $c_7 = 90$.

4. $e_3^- = 30$, $e_4^- + e_5^- = -30$, $e_{10}^- = -45$, $e_{11}^- = +45$.

5. $e_3^+ = 30$, $e_4^+ + e_5^+ = -30$, $e_{10}^+ = +45$, $e_{11}^+ = -45$, $e_{12}^+ = 180$, $e_{13}^+ = -180$.

Proof: Let $\mathcal{B}$ be a suitable boundary condition for an operator of Laplace type on M. Let $\alpha, \beta \in \mathbb{R}$. We define a time-dependent family of operators of Laplace type by setting

$$\mathfrak{D} := (1 + 2\alpha t + 3\beta t^2)D\,.$$

We set $\mathcal{B}_t = \mathcal{B}$ to keep the boundary condition static. By Lemma 3.2.6,

$$\begin{aligned} a_2(f, \mathfrak{D}, \mathfrak{B}) &= a_2(f, D, \mathcal{B}) - \tfrac{m}{2}\alpha a_0(f, D, \mathcal{B}), \\ a_3(f, \mathfrak{D}, \mathfrak{B}) &= a_3(f, D, \mathcal{B}) - \tfrac{m-1}{2}\alpha a_1(f, D, \mathcal{B}), \\ a_4(f, \mathfrak{D}, \mathfrak{B}) &= a_4(f, D, \mathcal{B}) - \tfrac{m-2}{2}\alpha a_2(f, D, \mathcal{B}) \\ &\quad + (\tfrac{m(m+2)}{8}\alpha^2 - \tfrac{m}{2}\beta)a_0(f, D, \mathcal{B})\,. \end{aligned} \tag{3.12.b}$$

We use Display (3.12.a) to study $a_n(f, \mathfrak{D}, \mathfrak{B})$ and we use Theorem 3.6.1 to study $a_{n-2}(f, D, \mathcal{B})$ and $a_{n-4}(f, D, \mathcal{B})$. The identities of Display (3.12.b) then permit us to derive a number of equations from which the Lemma will follow. Note that since m is arbitrary, equations involving this parameter can give rise to more than one relation.

We have the relations

$$\mathcal{G}_{1,ij} = -2\alpha\delta_{ij}, \quad \mathcal{F}_{1,i} = 0, \quad \mathcal{G}_{2,ij} = -3\beta\delta_{ij}, \quad \mathcal{E}_1 = -2\alpha E\,.$$

Studying a_2 gives rise to the equation

$$-(4\pi)^{-m/2}\tfrac{2}{6}mc_0\int_M(\alpha f)dx = -(4\pi)^{-m/2}\tfrac{m}{2}\int_M(\alpha f)dx$$

which yields Assertion (1). Studying a_3 gives rise to the equations:

$$-\tfrac{2}{384}\int_{C_N}\Big\{(m-1)e_1^+ + e_2^+\Big\}(\alpha f)dy = -\tfrac{m-1}{8}\int_{C_N}(\alpha f)dy,$$

$$-\tfrac{2}{384}\int_{C_D}\Big\{(m-1)e_1^- + e_2^-\Big\}(\alpha f)dy = \quad \tfrac{m-1}{8}\int_{C_D}(\alpha f)dy\,.$$

which yields Assertion (2). Examining the interior integrands in a_4 yields the following relationships from which Assertion (3) follows

$$\begin{aligned}
\tfrac{4}{360}(m^2c_1+mc_2)\int_M(\alpha^2 f)dx &= \tfrac{m(m+2)}{8}\int_M(\alpha^2 f)dx,\\
-\tfrac{3}{360}mc_3(\beta f)\int_M dx &= -\tfrac{m}{2}\int_M(\beta f)dx,\\
-\tfrac{2}{360}(c_4+mc_7)\int_M(\alpha f E)dx &= -\tfrac{m-2}{2}\int_M(\alpha f E)dx,\\
-\tfrac{2}{360}(mc_5+c_6)\int_M(\alpha f\tau)dx &= -\tfrac{m-2}{12}\int_M(\alpha f\tau)dx\,.
\end{aligned}$$

The boundary integrals over the Dirichlet component C_D show that

$$\begin{aligned}
-\tfrac{2}{360}\{(m-1)e_3^- + e_4^- + e_5^-\}\int_{C_D}(\alpha f L_{aa})dy &= -\tfrac{m-2}{6}\int_{C_D}(\alpha f L_{aa})dy,\\
-\tfrac{2}{360}\{(m-1)e_{10}^- + e_{11}^-\}\int_{C_D}(\alpha f_{;m})dy &= \tfrac{m-2}{4}\int_{C_D}(\alpha f_{;m})dy\,.
\end{aligned}$$

This establishes Assertion (4). Finally, we use the boundary integrals over the Robin component C_N to see

$$\begin{aligned}
-\tfrac{2}{360}\{(m-1)e_3^+ + e_4^+ + e_5^+\}\int_{C_N}(\alpha f L_{aa})dy &= -\tfrac{m-2}{6}\int_{C_N}(\alpha f L_{aa})dy,\\
-\tfrac{2}{360}\{(m-1)e_{10}^+ + e_{11}^+\}\int_{C_N}(\alpha f_{;m})dy &= -\tfrac{m-2}{4}\int_{C_N}(\alpha f_{;m})dy,\\
-\tfrac{2}{360}\{(m-1)e_{12}^+ + e_{13}^+\}\int_{C_N}(\alpha f S)dy &= -\tfrac{m-2}{1}\int_{C_N}(\alpha f S)dy\,.
\end{aligned}$$

We use these relations to complete the proof of the Lemma. □

Remark: The product formulae of Lemma 3.2.5 provide useful cross-checks on the values determined above but yield no additional information.

We make a time-dependent gauge change to establish the following result:

Lemma 3.12.3

1. $c_8 = 60$.
2. $e_{14}^+ - 2e_9^+ = 60$.
3. $e_9^- = -30$.

Proof: Let $D := -\partial_\theta^2$ on $M := [0,1]$ and let

$$\begin{aligned}
&D_{t,\varrho} := e^{-t\varrho\Psi}De^{t\varrho\Psi} + \varrho\Psi, && E = -\varrho\Psi,\\
&\mathcal{G}_{1,11} = 0, && \mathcal{G}_{2,11} = 0,\\
&\mathcal{F}_{1,1} = -2\varrho\Psi_{;\theta}, && \mathcal{E}_1 = -\varrho\Psi_{;\theta\theta}\,.
\end{aligned}$$

Choose $\mathfrak{B}_\varrho$ suitably. By Lemma 3.2.7,

$$\partial_\varrho a_n(f, \mathfrak{D}_\varrho, \mathfrak{B}_\varrho)|_{\varrho=0} = -a_{n-2}(f\Psi, D, \mathcal{B}) . \tag{3.12.c}$$

We set $n = 4$ and apply Theorem 3.6.1 to compute $a_4(f, D, \mathcal{B})$ and $a_2(f, D, \mathcal{B})$.

To study the variation of the interior integrands, we note

$$\partial_\varrho\{E_{;\theta\theta}\}|_{\varrho=0} = -\Psi_{;\theta\theta}, \quad \partial_\varrho\{\mathcal{E}_1\}|_{\varrho=0} = -\Psi_{;\theta\theta}, \quad \partial_\varrho\{\mathcal{F}_{1,i;i}\}|_{\varrho=0} = -2\Psi_{;\theta\theta} .$$

Since $\int_M (f\Psi_{;\theta\theta})dx$ does not appear in the interior integral for $a_2(f\Psi, D, \mathcal{B})$,

$$(-60 - c_4 - 2c_8) \int_M f\Psi_{;\theta\theta} dx = 0 .$$

Since $c_4 = -180$ by Lemma 3.12.2, $c_8 = 60$. This establishes Assertion (1).

To prove Assertion (2), we take Robin boundary conditions

$$\mathcal{B}_{t,\varrho}\phi := \{\phi_{;m} + S\phi + t\varrho\Psi_{;m}\phi\}|_{\partial M} .$$

We then have $S_1 = \varrho\Psi_{;m}$. The relevant variational formulae then become

$$\partial_\varrho\{E_{;m}\}|_{\varrho=0} = -\Psi_{;m}, \quad \partial_\varrho\{\mathcal{F}_{1,m}\}|_{\varrho=0} = -2\Psi_{;m}, \quad \partial_\varrho\{S_1\}|_{\varrho=0} = \Psi_{;m} .$$

Studying the boundary integral $\int_{\partial M}(f\Psi_{;m})dy$ then yields the identity

$$-240 - 2e_9^+ + e_{14}^+ = -180 .$$

Assertion (2) now follows. To prove Assertion (3), we take $\mathcal{B}$ to be the Dirichlet boundary operator. We use the invariant $\int_{\partial M} f\Psi_{;m}dy$ to see

$$120 - 2e_9^- = 180 .$$

The final assertion of the Lemma now follows. $\square$

We make a time-dependent change of coordinates to determine the remaining coefficients.

Lemma 3.12.4

1. $c_9 = 15$ *and* $c_{10} = -30$.
2. $e_4^- = -60$, $e_5^- = 30$, $e_6^- = 30$, $e_7^- = -30$, *and* $e_8^- = 0$.
3. $e_4^+ = 120$, $e_5^+ = -150$, $e_6^+ = -60$, $e_7^+ = 60$, $e_8^+ = 0$, $e_9^+ = 150$, $e_{14}^+ = 360$, *and* $e_{15}^+ = 0$.

Proof: We apply Lemma 3.2.8. Let $M := S^1 \times [0, 1]$ be given the metric

$$ds^2 := e^{2\varepsilon\psi_1}dx_1^2 + e^{2\varepsilon\psi_2}dx_2^2$$

where the functions ψ_i are smooth on M. The volume form becomes

$$dx = g dx_1 dx_2 \quad \text{where} \quad g := e^{\varepsilon(\psi_1+\psi_2)} .$$

Let $\psi_{i/j} := \partial_j\psi_i$, $\psi_{i/jk} := \partial_j\partial_k\psi_i$ and so forth. The scalar Laplacian can then be represented in the form

$$\begin{aligned}\Delta = -g^{-1}\partial_i g g^{ij}\partial_j = &-\{e^{-2\varepsilon\psi_1}\partial_1^2 + e^{-2\varepsilon\psi_2}\partial_2^2 \\ &+\varepsilon(\psi_{2/1} - \psi_{1/1})\partial_1 + \varepsilon(\psi_{1/2} - \psi_{2/2})\partial_2\} + O(\varepsilon^2) .\end{aligned}$$

Let Ξ be an auxiliary function of compact support on M. We define a diffeomorphism Φ_ϱ of a neighborhood of $M \times \{0\}$ in $M \times [0, \infty)$ by setting

$$\Phi_\varrho(x_1, x_2; t) := (t, x_1 + \varrho t \Xi(x_1, x_2), x_2)\,.$$

Let $\Xi_{/i} = \partial_i \Xi$, $\Xi_{/ij} := \partial_i \partial_j \Xi$ and so forth. We then have that

$$\begin{aligned}
&\Phi_\varrho^* dx^1 = dx^1 + \varrho(\Xi dt + t\Xi_{/1} dx^1 + t\Xi_{/2} dx^2),\\
&\Phi_\varrho^*(\partial_1) = \partial_1 - t\varrho \Xi_{/1}\partial_1,\\
&\Phi_\varrho^* dx^2 = dx^2,\\
&\Phi_\varrho^*(\partial_2) = \partial_2 - t\varrho\Xi_{/2}\partial_1,\\
&\Phi_\varrho^* dt = dt,\\
&\Phi_\varrho^*(\partial_t) = \partial_t - \varrho\Xi\partial_1\,.
\end{aligned}$$

We suppress terms which are $O(\varepsilon^2) + O(\varrho^2)$ and expand in a Taylor series

$$\begin{aligned}
\Phi_\varrho^* e^{-2\varepsilon\psi_1} &= e^{-2\varepsilon\psi_1} - 2\varepsilon\varrho t\Xi\psi_{1/1} + ...,\\
\Phi_\varrho^* e^{-2\varepsilon\psi_2} &= e^{-2\varepsilon\psi_2} - 2\varepsilon\varrho t\Xi\psi_{2/1} + ...,\\
\Phi_\varrho^* \varepsilon(\psi_{2/1} - \psi_{1/1}) &= \varepsilon(\psi_{2/1} - \psi_{1/1}) + \varepsilon\varrho t\Xi(\psi_{2/11} - \psi_{1/11}) + ...,\\
\Phi_\varrho^* \varepsilon(\psi_{1/2} - \psi_{2/2}) &= \varepsilon(\psi_{1/2} - \psi_{2/2}) + \varepsilon\varrho t\Xi(\psi_{1/21} - \psi_{2/21}) + ...
\end{aligned}$$

This implies that

$$\begin{aligned}
\Phi_\varrho^*\{-e^{-2\varepsilon\psi_1}\partial_1^2\} &= -e^{-2\varepsilon\psi_1}\partial_1^2 + 2t\varepsilon\varrho\Xi\psi_{1/1}\partial_1^2\\
&\quad + t\varrho e^{-2\varepsilon\psi_1}\{\Xi_{/11}\partial_1 + 2\Xi_{/1}\partial_1^2\} + ...,\\
\Phi_\varrho^*\{-e^{-2\varepsilon\psi_2}\partial_2^2\} &= -e^{-2\varepsilon\psi_2}\partial_2^2 + 2t\varepsilon\varrho\Xi\psi_{2/1}\partial_2^2\\
&\quad + t\varrho e^{-2\varepsilon\psi_2}\{\Xi_{/22}\partial_1 + 2\Xi_{/2}\partial_1\partial_2\} + ...,\\
\varepsilon\Phi_\varrho^*\{-(\psi_{2/1} - \psi_{1/1})\partial_1\} &= \varepsilon\{-\psi_{2/1} + \psi_{1/1} - \varrho t\Xi(\psi_{2/11} - \psi_{1/11})\\
&\quad + t\varrho\Xi_{/1}(\psi_{2/1} - \psi_{1/1})\}\partial_1 + ...,\\
\varepsilon\Phi_\varrho^*\{-(\psi_{1/2} - \psi_{2/2})\partial_2\} &= \varepsilon\{-\psi_{1/2} + \psi_{2/2} - \varrho t\Xi(\psi_{1/21} - \psi_{2/21})\}\partial_2\\
&\quad + t\varepsilon\varrho\Xi_{/2}(\psi_{1/2} - \psi_{2/2})\partial_1 + ...
\end{aligned}$$

Consequently, we have that

$$\begin{aligned}
\mathfrak{D}_\varrho \quad &:= \quad \Phi_\varrho^*(\partial_t + \Delta) - \partial_t = \Delta - \varrho\Xi\partial_1\\
&+ \quad t\varrho\{e^{-2\varepsilon\psi_1}(2\Xi_{/1}\partial_1^2 + \Xi_{/11}\partial_1)\\
&+ \quad e^{-2\varepsilon\psi_2}(2\Xi_{/2}\partial_1\partial_2 + \Xi_{/22}\partial_1)\}\\
&+ \quad t\varepsilon\varrho\{2\psi_{1/1}\Xi\partial_1^2 + 2\psi_{2/1}\Xi\partial_2^2\\
&+ \quad [(\psi_{2/1} - \psi_{1/1})\Xi_{/1} - (\psi_{2/11} - \psi_{1/11})\Xi\\
&+ \quad (\psi_{1/2} - \psi_{2/2})\Xi_{/2}]\partial_1 - (\psi_{1/12} - \psi_{2/12})\Xi\partial_2\} + ...
\end{aligned}$$

Since the coordinate frame is not orthonormal, we must be careful with which indices are up and which indices are down. Let ω be the connection 1 form determined by D_0. Again, we omit terms which are $O(\varepsilon^2) + O(\varrho^2)$. The previous Equation may then be used to see that

$$\begin{array}{ll} D_0 = \Delta - \varrho\Xi\partial_1, & \omega_1 = \frac{1}{2}e^{2\varepsilon\psi_1}\varrho\Xi + ..., \\ \mathcal{G}_{1,}{}^{11} = e^{-2\varepsilon\psi_1}2\varrho\Xi_{/1} + 2\varepsilon\varrho\psi_{1/1}\Xi + ..., & \omega_2 = 0 + ..., \\ \mathcal{G}_{1,}{}^{22} = 2\varepsilon\varrho\psi_{2/1}\Xi + ..., & \mathcal{G}_{1,}{}^{12} = e^{-2\varepsilon\psi_2}\varrho\Xi_{/2} + ..., \\ E = -\frac{1}{2}\varrho\Xi_{/1} - \frac{1}{2}\varepsilon\varrho(\psi_{1/1} + \psi_{2/1})\Xi + ..., & \mathcal{E}_1 = 0 + ... \end{array}$$

To compute $\mathcal{F}$, we must express partial differentiation in terms of covariant differentiation. Since ω is linear in ϱ, it plays no role. The Christoffel symbols of the metric, however, play a crucial role. Relative to the coordinate frame,

$$\begin{aligned} \Gamma_{121} = \Gamma_{211} = -\Gamma_{112} &= \tfrac{1}{2}\varepsilon\psi_{1/2} + O(\varepsilon^2), \\ \Gamma_{212} = \Gamma_{122} = -\Gamma_{221} &= \tfrac{1}{2}\varepsilon\psi_{2/1} + O(\varepsilon^2), \\ \Gamma_{111} &= \tfrac{1}{2}\varepsilon\psi_{1/1} + O(\varepsilon^2), \\ \Gamma_{222} &= \tfrac{1}{2}\varepsilon\psi_{2/2} + O(\varepsilon^2) . \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{G}_{1,}{}^{11}f_{;11} &= (\mathcal{G}_{1,}{}^{11}\partial_1^2 - 2\varepsilon\varrho\psi_{1/1}\Xi_{/1}\partial_1 + 2\varepsilon\varrho\psi_{1/2}\Xi_{/1}\partial_2)f + ... \\ 2\mathcal{G}_{1,}{}^{12}f_{;12} &= (2\mathcal{G}_{1,}{}^{12}\partial_1\partial_2 - 2\varepsilon\varrho\psi_{1/2}\Xi_{/2}\partial_1 - 2\varepsilon\varrho\psi_{2/1}\Xi_{/2}\partial_2)f + ... \\ \mathcal{G}_{1,}{}^{22}f_{;22} &= \mathcal{G}_{1,}{}^{22}\partial_2^2 f + ... \end{aligned}$$

We use this computation to determine the tensor $\mathcal{F}_1$:

$$\begin{aligned} \mathcal{F}_{1,}{}^1 &= \varrho(e^{-2\varepsilon\psi_1}\Xi_{/11} + e^{-2\varepsilon\psi_2}\Xi_{/22}) \\ &+ \varepsilon\varrho\{(\psi_{2/1} - \psi_{1/1})\Xi_{/1} - (\psi_{2/11} - \psi_{1/11})\Xi \\ &+ (\psi_{1/2} - \psi_{2/2})\Xi_{/2} + 2\psi_{1/1}\Xi_{/1} + 2\psi_{1/2}\Xi_{/2}\}, \\ \mathcal{F}_{1,}{}^2 &= \varepsilon\varrho\{-(\psi_{1/12} - \psi_{2/12})\Xi - 2\psi_{1/2}\Xi_{/1} + 2\psi_{2/1}\Xi_{/2}\} + ... \end{aligned}$$

Let $\mathcal{B}_{t,\varrho} := \Phi_\varrho^*\mathcal{B}_t$. By Lemma 3.2.8,

$$\partial_\varrho a_n(f, \mathfrak{D}_\varrho, \mathfrak{B})|_{\varrho=0} = -\tfrac{1}{2}a_{n-2}(g^{-1}\partial_1(gf\Xi), \Delta, \mathcal{B}) .$$

To prove Assertion (1) of the Lemma, let $P \in \operatorname{int}(M)$ and let

$$\psi_1(P) = \psi_2(P) = 0 .$$

We study monomials $\Xi_{/111}$ and $\psi_{2/111}\Xi$ appearing in the interior integrals for $\partial_\varrho|_{\varrho=0}a_4$. Let $\mathcal{R} = E$ or let $\mathcal{R} = \tau$. We integrate by parts to define $\mathcal{A}[\mathcal{R}]$ by the identity:

$$-\tfrac{1}{12}\int_M g^{-1}\partial_1(gf\Xi)\mathcal{R}dx = \tfrac{1}{360}\int_M f\mathcal{A}[\mathcal{R}]dx .$$

One then has that

$$-\tfrac{1}{2}a_2^M(g^{-1}\partial_1(gf\Xi), \Delta) = (4\pi)^{-1}\tfrac{1}{360}\int_M f\mathcal{A}[6E + \tau]dx .$$

We have $\tau \equiv -2\varepsilon\psi_{2/11} + \ldots$. We compute

$$\begin{array}{lcrr} \partial_\varrho|_{\varrho=0}60E_{;ii} & = & -30\Xi_{/111} & -30\varepsilon\psi_{2/111}\Xi + \ldots, \\ \partial_\varrho|_{\varrho=0}60\mathcal{F}_{1,i;i} & = & 60\Xi_{/111} & -60\varepsilon\psi_{2/111}\Xi + \ldots, \\ \partial_\varrho|_{\varrho=0}c_9\mathcal{G}_{1,ii;jj} & = & 2c_9\Xi_{/111} & +2c_9\varepsilon\psi_{2/111}\Xi + \ldots, \\ \partial_\varrho|_{\varrho=0}c_{10}\mathcal{G}_{1,ij;ij} & = & 2c_{10}\Xi_{/111} & +0c_{10}\varepsilon\psi_{2/111}\Xi + \ldots, \\ \mathcal{A}[6E] & = & 0\Xi_{/111} & +0\varepsilon\psi_{2/111}\Xi + \ldots, \\ \mathcal{A}[\tau] & = & 0\Xi_{/111} & -60\varepsilon\psi_{2/111}\Xi + \ldots. \end{array}$$

Assertion (1) follows since one has

$$\begin{aligned} &-30 + 60 + 2c_9 + 2c_{10} = 0, \quad \text{and} \\ &-30 - 60 + 2c_9 = -60\,. \end{aligned}$$

We now study the boundary terms. We pull back the Robin boundary operator $\mathcal{B} := e^{-\varepsilon\psi_2}\partial_2 + S$ to express

$$\begin{aligned} &\Phi_\varrho^*\mathcal{B} = e^{-\varepsilon\psi_{2/1}t\varrho\Xi}\{\mathcal{B} - e^{-\varepsilon\psi_2}t\varrho\Xi_{/2}\partial_1 + t\varrho\Xi(\varepsilon S\psi_{2/1} + S_{/1})\} + \ldots, \\ &\Gamma_1 = -e^{-\varepsilon\psi_2}\varrho\Xi_{/2} \quad \text{and} \quad S_1 = \varrho\Xi(\varepsilon S\psi_{2/1} + S_{/1}) + \ldots\,. \end{aligned}$$

With Dirichlet boundary conditions, we set $S = \Gamma = 0$. We study the terms that make up $\partial_\varrho|_{\varrho=0}a_4(f, \mathfrak{D}_\varrho, \mathcal{B}_\varrho)$, focusing on the boundary contributions. Let $\equiv$ indicate we are working modulo $O(\varepsilon^2)$. Thus, for example, we have that $L_{11} \equiv -\varepsilon\psi_{1/2}$. At the point of the boundary in question, we suppose $\psi_1(P) = \psi_2(P) = 0$. We list the coefficient as $-$ for Dirichlet and $+$ for Robin boundary conditions.

$$\begin{aligned} &\partial_\varrho|_{\varrho=0}(-120^-, 240^+)fE_{;m} \\ &\qquad \equiv (60^-, -120^+)f\{\Xi_{/12} + (\varepsilon\psi_{1/12} + \varepsilon\psi_{2/12})\Xi \\ &\qquad\qquad + (\varepsilon\psi_{1/1} + \varepsilon\psi_{2/1})\Xi_{/2}\}, \\ &\partial_\varrho|_{\varrho=0}120fEL_{aa} \equiv 60\varepsilon f\psi_{1/2}\Xi_{/1}, \\ &\partial_\varrho|_{\varrho=0}720fSE \equiv -360fS\{\Xi_1 + \varepsilon(\psi_{1/1} + \psi_{2/1})\Xi\}, \\ &\partial_\varrho|_{\varrho=0}e_3^\pm f\mathcal{G}_{1,aa}L_{bb} \equiv e_3^\pm f(2\Xi_{/1})(-\varepsilon\psi_{1/2}), \\ &\partial_\varrho|_{\varrho=0}e_4^\pm f\mathcal{G}_{1,mm}L_{bb} \equiv 0, \\ &\partial_\varrho|_{\varrho=0}e_5^\pm f\mathcal{G}_{1,ab}L_{ab} \equiv e_5^\pm f(2\Xi_{/1})(-\varepsilon\psi_{1/2}), \\ &\partial_\varrho|_{\varrho=0}e_6^\pm f\mathcal{G}_{1,mm;m} \equiv e_6^\pm f(2\varepsilon\psi_{2/12}\Xi + 4\varepsilon\psi_{2/1}\Xi_{/2}), \\ &\partial_\varrho|_{\varrho=0}e_7^\pm f\mathcal{G}_{1,aa;m} \equiv e_7^\pm f\{2\Xi_{/12} + 2\varepsilon\psi_{1/12}\Xi + 2\varepsilon\psi_{1/1}\Xi_{/2} - 2\varepsilon\psi_{2/1}\Xi_{/2}\}, \\ &\partial_\varrho|_{\varrho=0}e_8^\pm f\mathcal{G}_{1,am;a} \equiv e_8^\pm f\{-\varepsilon\psi_{2/1}\Xi_{/2} + \Xi_{/12} + \varepsilon\psi_{1/1}\Xi_{/2} - 2\varepsilon\psi_{1/2}\Xi_{/1}\}, \\ &\partial_\varrho|_{\varrho=0}e_9^\pm f\mathcal{F}_{1,m} \equiv e_9^\pm f\{-(\varepsilon\psi_{1/12} - \varepsilon\psi_{2/12})\Xi - 2\varepsilon\psi_{1/2}\Xi_{/1} + 2\varepsilon\psi_{2/1}\Xi_{/2}\}, \\ &\partial_\varrho|_{\varrho=0}e_{12}^+ fS\mathcal{G}_{1,aa} \equiv e_{12}^+ f\{2\Xi_{/1}S + 2\varepsilon\psi_{1/1}\Xi S\}, \\ &\partial_\varrho|_{\varrho=0}e_{13}^+ fS\mathcal{G}_{1,mm} \equiv e_{13}^+ f\{2\varepsilon\psi_{2/1}\Xi S\}, \\ &\partial_\varrho|_{\varrho=0}e_{14}^+ fS_1 \equiv e_{14}^+ f\Xi\{\varepsilon\psi_{2/1}S + S_{/1}\}, \\ &\partial_\varrho|_{\varrho=0}e_{15}^+ f\Gamma_{1,a:a} \equiv e_{15}^+ f(\varepsilon\psi_{2/1}\Xi_{/2} - \Xi_{/12} - \varepsilon\psi_{1/1}\Xi_{/2}), \end{aligned}$$

$$\partial_\varrho|_{\varrho=0}(\pm 180) f_{;m} E \equiv \mp 90 f_{;m}\{\Xi_{/1} + (\varepsilon\psi_{1/1} + \varepsilon\psi_{2/1})\Xi\},$$
$$\partial_\varrho|_{\varrho=0} e_{10}^{\pm} f_{;m} \mathcal{G}_{1,aa} \equiv e_{10}^{\pm} f_{;m}(2\Xi_{/1} + 2\varepsilon\psi_{1/1}\Xi),$$
$$\partial_\varrho|_{\varrho=0} e_{11}^{\pm} f_{;m} \mathcal{G}_{1,mm} \equiv e_{11}^{\pm} f_{;m} 2\varepsilon\psi_{2/1}\Xi\,.$$

We must also study the boundary terms comprising $-\frac{1}{2}a_2^{\partial M}(\cdot)$. As when studying a_2^M, we integrate by parts to define $\mathcal{A}$ and compute:

$$\mathcal{A}[2fL_{aa}] \equiv -60\varepsilon f\psi_{1/12}\Xi,$$
$$\mathcal{A}[12fS] \equiv -360\{\Xi\varepsilon fS\psi_{2/1} - f\Xi S_{/1}\},$$
$$\mathcal{A}[\pm 3f_{;m}] \equiv \mp 90\{(\varepsilon\psi_{1/12} + \varepsilon\psi_{2/12})f\Xi + 2\varepsilon\psi_{2/1}(f_{;m}\Xi + f\Xi_{/2})\}.$$

We established the following relations in Lemmas 3.12.2 and 3.12.3

$$e_3^{\pm} = 30,\ e_4^{\pm} + e_5^{\pm} = -30,\ e_{14}^{+} - 2e_9^{+} = 60 \text{ and } e_9^{-} = -30\,.$$

We use Lemma 3.2.8 to derive the following equations and complete the proof, the relevant monomial being listed in the first column.

monomial	$\varrho a_4(\cdot)\vert_{\varrho=0}$	$= -\frac{1}{2}a_{n-2}(\cdot)$
$f\varepsilon\psi_{2/1}\Xi_{/2}$	$(60^-, -120^+) + 4e_6^{\pm} - 2e_7^{\pm} - e_8^{\pm} + 2e_9^{\pm} + e_{15}^{\pm}$	$= \mp 180,$
$f\varepsilon\psi_{2/12}\Xi$	$(60^-, -120^+) + 2e_6^{\pm} + e_9^{\pm}$	$= \mp 90,$
$f\varepsilon\psi_{1/12}\Xi$	$(60^-, -120^+) + 2e_7^{\pm} - e_9^{\pm}$	$= -60 \mp 90,$
$f\Xi_{/12}$	$(60^-, -120^+) + 2e_7^{\pm} + e_8^{\pm} - e_{15}^{\pm}$	$= 0,$
$f\varepsilon\psi_{1/2}\Xi_{/1}$	$-2e_5^{\pm} - 2e_8^{\pm} - 2e_9^{\pm}$	$= 0,$
$fS_{/1}\Xi$	e_{14}^{+}	$= 360,$
$f\varepsilon\psi_{2/1}\Xi S$	$-360 + 2e_{13}^{+} + e_{14}^{+}$	$= -360,$
$f\Xi_{/1}S$	$-360 + 2e_{12}^{+}$	$= 0,$
$f_{;m}\varepsilon\psi_{2/1}\Xi$	$\mp 90 + 2e_{11}^{\pm}$	$= \mp 180,$
$f_{;m}\Xi_{/1}$	$\mp 90 + 2e_{10}^{\pm}$	$= 0\,.$

This completes the proof of Lemma 3.12.4 and thereby also the proof of Theorem 3.12.1. □

Remark 3.12.5 One can use Lemma 3.2.9 to derive additional relations providing a useful cross-check of the computations performed above. The redundancy of the computational algorithm provides a useful amount of robustness in helping to guard against calculational mistakes!

3.13 The eta invariant

In the next section, we will study spectral boundary conditions. Theorem 3.2.14 expresses the boundary correction terms to the heat trace asymptotics in terms of the invariants a_k^η if the structures are product near the boundary. In this section, we present results of Branson and Gilkey [85, 86] concerning these invariants.

Let M be a compact Riemannian manifold. Let P be an operator of Dirac type. If the boundary of M is non-empty, then we must impose suitable boundary conditions. Suppose there exists an endomorphism χ of $V|_{\partial M}$ so that χ is self-adjoint and so that

$$\chi^2 = 1, \quad \chi\gamma_m + \gamma_m\chi = 0, \quad \text{and} \quad \chi\gamma_a = \gamma_a\chi\,.$$

We let $\mathcal{B}\phi := \Pi_\chi^-\phi|_{\partial M}$. The associated boundary condition for $D := P^2$ is defined by the boundary operator

$$\mathcal{B}_1\phi := \mathcal{B}\phi \oplus \mathcal{B}P\phi$$

and is equivalent to a mixed boundary operator $\mathcal{B}_{\chi,S}$ where S is given in Display (3.13.a) below.

Let $D = P^2$ be the associated operator of Laplace type. Let F be an endomorphism of the bundle. As $t \downarrow 0$, there is an asymptotic expansion

$$\mathrm{Tr}_{L^2}(FPe^{-tP_{\mathcal{B}}^2}) \sim \sum_{n=0}^\infty a_n^\eta(F,P,\mathcal{B})t^{(n-m-1)/2}$$

where the indexing convention is chosen so that the local invariants a_n^η are given by local formulae that are homogeneous of total weight n in the jets of the symbol of P. These invariants measure the spectral asymmetry of P;

$$a_n^\eta(F,P,\mathcal{B}) = -a_n^\eta(F,-P,\mathcal{B})\,.$$

Theorem 3.13.1 *Let M be a compact Riemannian manifold of dimension m with smooth boundary. Let γ define a Clifford module structure on a bundle V over M. Let ∇ be a compatible connection and let $P = \gamma^i\nabla_{e_i} + \psi_P$ be an operator of Dirac type on V. Let $W_{ij} := \Omega_{ij} - \frac{1}{4}R_{ijkl}\gamma_k\gamma_l$ where Ω is the curvature of ∇. Let $f \in C^\infty(M)$. Then:*

1. $a_0^\eta(f,P,\mathcal{B}) = 0$.
2. $a_1^\eta(f,P,\mathcal{B}) = -(4\pi)^{-m/2}(m-1)\int_M \mathrm{Tr}\,\{f\psi_P\}dx$.
3. $a_2^\eta(f,P,\mathcal{B}) = \frac{1}{4}(4\pi)^{-(m-1)/2}\int_{\partial M}(2-m)\mathrm{Tr}\,\{f\psi_P\chi\}dy$.
4. $$\begin{aligned} a_3^\eta(f,P,\mathcal{B}) = &-\tfrac{1}{12}(4\pi)^{-m/2}\int_M f\big\{\mathrm{Tr}\,\{2(m-1)\nabla_{e_i}\psi_P + 3(4-m)\psi_P\gamma_i\psi_P \\ &+3\gamma_j\psi_P\gamma_j\gamma_i\psi_P\}_{;i} + (m-3)\mathrm{Tr}\,\{-\tau\psi_P - 6\gamma_i\gamma_jW_{ij}\psi_P + 6\gamma_i\psi_P\nabla_{e_i}\psi_P \\ &+(m-4)\psi_P^3 - 3\psi_P^2\gamma_j\psi_P\gamma_j\}\big\}dx \\ &-\tfrac{1}{12}(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\,\{6(m-2)f_{;m}\chi\psi_P + f[(6m-18)\chi\nabla_{e_m}\psi_P \\ &+2(m-1)\nabla_{e_m}\psi_P + 6\chi\gamma_m\gamma_a\nabla_{e_a}\psi_P + 6(2-m)\chi\psi_PL_{aa} \\ &+2(3-m)\psi_PL_{aa} + 6(3-m)\chi\gamma_m\psi_P^2 + 3\gamma_m\psi_P\gamma_a\psi_P\gamma_a \\ &+3(3-m)\chi\gamma_m\psi_P\chi\psi_P + 6\chi\gamma_aW_{am}]\}dy. \end{aligned}$$

Remark 3.13.2 We have changed the sign convention for ψ_P from that employed in [85] in the interest of making the formulae in this section compatible with formulae in the next section. We emphasize that ∇ is in general **not** the connection ∇^D.

Proof: One has by [85, 86], that

$$\begin{aligned}
&\nabla^D_{e_i} = \nabla_{e_i} + \theta_i \quad \text{where} \quad \theta_i := -\tfrac{1}{2}(\psi_P\gamma_i + \gamma_i\psi_P),\\
&E = \tfrac{1}{2}(\nabla_{e_i}\psi_P\gamma_i - \gamma_i\nabla_{e_i}\psi_P) - \psi_P^2 - \theta_i\theta_i - \tfrac{1}{2}\gamma_i\gamma_j W_{ij} - \tfrac{1}{4}\tau,\\
&S = \tfrac{1}{2}\Pi_+(\gamma_m\psi_P - \psi_P\gamma_m - L_{aa}\chi)\Pi_+\,.
\end{aligned} \qquad (3.13.a)$$

Let $P_\varepsilon := P - \varepsilon f$. Lemma 3.2.11 shows that

$$\partial_\varepsilon a_n(1, (P-\varepsilon f)^2, \mathcal{B}_1)|_{\varepsilon=0} = 2a^\eta_{n-1}(f, P, \mathcal{B})\,. \qquad (3.13.b)$$

The Theorem now follows, after bit of calculation, from these relations, from Equation (3.13.b), and from Theorem 3.3.1; the metric contributions are unchanged. We shall omit the details in the interest of brevity and refer instead to Branson and Gilkey [85, 86]. □

As we discussed in Section 1.6.6, if m is odd, then P may not admit suitable local boundary conditions. Thus it is natural to impose spectral boundary conditions as shall be discussed presently in Section 3.14. Lemma 3.2.11 together with Theorem 3.14.1 could then be used to analyze the boundary contribution for a^η_n in this setting for $n \le 2$. We omit details in the interest of brevity.

3.14 Spectral boundary conditions

Let M be a compact Riemannian manifold of dimension m with smooth boundary. Let V_i be smooth vector bundles over M which are equipped with smooth fiber metrics. Let $P : C^\infty(V_1) \to C^\infty(V_2)$ define an *elliptic complex of Dirac type*; this means that the associated second order operators

$$D_1 := P^*P \qquad \text{and} \qquad D_2 := PP^*$$

are of Laplace type.

Let γ be the leading symbol of P. Then

$$\begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}$$

defines a *unitary Clifford module structure* on $V_1 \oplus V_2$. By Lemma 1.1.7, we may choose a unitary connection ∇ on $V_1 \oplus V_2$ which is compatible with respect to this Clifford module structure. By averaging ∇ with respect to the *chirality endomorphism*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we can assume that ∇ respects the splitting and induces connections ∇_1 and ∇_2 on the bundles V_1 and V_2, respectively.

Decompose P in the form

$$P = \gamma_i \nabla_{e_i} + \psi$$

where $\psi \in \text{End}(V_1, V_2)$. Near the boundary, the structures depend on the normal variable. We set the normal variable x_m to zero to define a tangential operator of Dirac type

$$B(y) := \gamma_m(y,0)^{-1}\left(\gamma_a(y,0)\nabla_{e_a} + \psi(y,0)\right) \text{ on } C^\infty(V_1|_{\partial M}).$$

Let B^* be the adjoint of B on $L^2(V_1|_{\partial M})$; Lemma 1.4.15 then generalizes to show

$$B^* = \gamma_m(y,0)^{-1}\gamma_a(y,0)\nabla_{e_a} + \psi_B^*$$

where $\psi_B := \gamma_m(y,0)^{-1}\psi(y,0)$. Let Θ be an auxiliary self-adjoint endomorphism of V_1. We set

$$\begin{aligned} A &:= \tfrac{1}{2}(B + B^*) + \Theta \quad \text{on} \quad C^\infty(V_1|_{\partial M}), \\ A^\# &:= -\gamma^m A(\gamma^m)^{-1} \quad \text{on} \quad C^\infty(V_2|_{\partial M}). \end{aligned}$$

The leading symbol of A is then given by

$$\gamma_a^T := \gamma_m^{-1}\gamma_a$$

which is a unitary Clifford module structure on $V_1|_{\partial M}$. Thus A is a self-adjoint operator of Dirac type on $C^\infty(V_1|_{\partial M})$; similarly $A^\#$ is a self-adjoint operator of Dirac type on $C^\infty(V_2|_{\partial M})$.

Let Π_A^+ (resp. $\Pi_{A\#}^+$) be spectral projection on the eigenspaces of A (resp. $A^\#$) corresponding to the positive (resp. non-negative) eigenvalues; there is always a bit of technical fuss concerning the harmonic eigenspaces that we will ignore as it does not affect the heat trace asymptotic coefficients that we shall be considering. Introduce the associated spectral boundary operators by

$$\begin{aligned} \mathcal{B}_1\phi_1 &:= \Pi_A^+(\phi_1|_{\partial M}) && \text{for} \quad \phi_1 \in C^\infty(V_1), \\ \mathcal{B}_2\phi_2 &:= \Pi_{A\#}^+(\phi_2|_{\partial M}) && \text{for} \quad \phi_2 \in C^\infty(V_2), \\ \mathcal{B}_s\phi_1 &:= \mathcal{B}_1\phi_1 \oplus \mathcal{B}_2(P\phi_1) && \text{for} \quad \phi_1 \in C^\infty(V_1). \end{aligned}$$

If $P_{\mathcal{B}_1}$, $P_{\mathcal{B}_2}^*$, and $D_{1,\mathcal{B}}$ are the realizations of P, of P^*, and of D_1 with respect to the boundary conditions $\mathcal{B}_1$,$\mathcal{B}_2$, and $\mathcal{B}_s$, respectively, then

$$(P_{\mathcal{B}_1})^* = P_{\mathcal{B}_2}^* \qquad \text{and} \qquad D_{1,\mathcal{B}_s} = P_{\mathcal{B}_1}^* P_{\mathcal{B}_1}.$$

The argument proving Lemma 1.6.7 extends immediately to this situation to show that $(D_1, \mathcal{B}_s)$ is self-adjoint and elliptic with respect to the cone $\mathcal{C}$.

It is worth putting this in the framework considered previously. Suppose $P = -\gamma_i\nabla_{e_i} + \psi_P$ is an operator of Dirac type on a vector bundle V where ∇ is a compatible connection. We suppose for the sake of simplicity that P is formally self-adjoint or, equivalently, that $\gamma^* = -\gamma$ and $\psi_P^* = \psi_P$. We take $V_1 = V_2 = V$ and set

$$B = -\gamma_m\gamma_a\nabla_{e_a} - \gamma_m\psi_P.$$

Then by Lemma 1.4.15,

$$\begin{aligned} B^* &= -\gamma_m\gamma_a\nabla_{e_a} + \psi_P\gamma_m, \\ A &= -\gamma_m\gamma_a\nabla_{e_a} + \tfrac{1}{2}(\psi_P\gamma_m - \gamma_m\psi_P) + \Theta, \\ \psi_A &= \tfrac{1}{2}(\psi_P\gamma_m - \gamma_m\psi_P) + \Theta. \end{aligned}$$

Thus the formalism of this section for the boundary condition is essentially equivalent to that used previously. It is convenient, however, to work in this new framework both because problems of this type arise in the study of index theory and also because it simplifies the invariance theory which is involved.

By Theorem 1.4.6, we have an asymptotic series

$$\operatorname{Tr}_{L^2}(fe^{-tD_{1,\mathcal{B}_s}}) \sim \sum_{k=0}^{m-1} a_k(f,D_1,\mathcal{B}_s)t^{(k-m)/2} + O(t^{-1/8})\,.$$

Define:

$$\begin{aligned}&\gamma_a^T := \gamma_m^{-1}\gamma_a, \quad \hat{\psi} := \gamma_m^{-1}\psi, \quad \text{and}\\ &\beta(m) := \Gamma(\tfrac{m}{2})\Gamma(\tfrac{1}{2})^{-1}\Gamma(\tfrac{m+1}{2})^{-1}\,.\end{aligned}$$

We refer to [144, 193] for the proof of the following result:

Theorem 3.14.1 *Let $f \in C^\infty(M)$. Then:*

1. $a_0(f,D_1,\mathcal{B}_s) = (4\pi)^{-m/2}\int_M \operatorname{Tr}(f\mathrm{Id})dx$.
2. *If $m \geq 2$, then* $a_1(f,D_1,\mathcal{B}_s) = \frac{1}{4}[\beta(m)-1](4\pi)^{-(m-1)/2}\int_{\partial M}\operatorname{Tr}(f\mathrm{Id})dy$.
3. *If $m \geq 3$, then* $a_2(f,D_1,\mathcal{B}_s) = (4\pi)^{-m/2}\int_M \frac{1}{6}\operatorname{Tr}\{f(\tau\mathrm{Id}+6E)\}dx$

$$+(4\pi)^{-m/2}\int_{\partial M}\operatorname{Tr}\{\tfrac{1}{2}[\hat{\psi}+\hat{\psi}^*]f + \tfrac{1}{3}[1-\tfrac{3}{4}\pi\beta(m)]L_{aa}f\mathrm{Id}$$
$$-\tfrac{m-1}{2(m-2)}[1-\tfrac{1}{2}\pi\beta(m)]f_{;m}\mathrm{Id}\}dy.$$

4. *If $m \geq 4$, then*

$$\begin{aligned}
a_3(f,D_1,\mathcal{B}) &= (4\pi)^{-(m-1)/2}\int_{\partial M}\operatorname{Tr}\{\tfrac{1}{32}(1-\tfrac{\beta(m)}{m-2})f(\hat{\psi}\hat{\psi}+\hat{\psi}^*\hat{\psi}^*)\\
&+\tfrac{1}{16}(5-2m+\tfrac{7-8m+2m^2}{m-2}\beta(m))f\hat{\psi}\hat{\psi}^* - \tfrac{1}{48}(\tfrac{m-1}{m-2}\beta(m)-1)f\tau\mathrm{Id}\\
&+\tfrac{1}{32(m-1)}(2m-3-\tfrac{2m^2-6m+5}{m-2}\beta(m))f(\gamma_a^T\hat{\psi}\gamma_a^T\hat{\psi}+\gamma_a^T\hat{\psi}^*\gamma_a^T\hat{\psi}^*)\\
&+\tfrac{1}{16(m-1)}(1+\tfrac{3-2m}{m-2}\beta(m))f\gamma_a^T\hat{\psi}\gamma_a^T\hat{\psi}^* + \tfrac{1}{48}(1-\tfrac{4m-10}{m-2}\beta(m))f\rho_{mm}\mathrm{Id}\\
&+\tfrac{1}{48(m+1)}(\tfrac{17+5m}{4}+\tfrac{23-2m-4m^2}{m-2}\beta(m))fL_{ab}L_{ab}\mathrm{Id}\\
&+\tfrac{1}{48(m^2-1)}(-\tfrac{17+7m^2}{8}+\tfrac{4m^3-11m^2+5m-1}{m-2}\beta(m))fL_{aa}L_{bb}\mathrm{Id}\\
&+\tfrac{1}{8(m-2)}\beta(m)f(\Theta\Theta+\tfrac{1}{m-1}\gamma_a^T\Theta\gamma_a^T\Theta)\} + \tfrac{m-1}{16(m-3)}(2\beta(m)-1)f_{;mm}\mathrm{Id}\\
&+\tfrac{1}{8(m-3)}(\tfrac{5m-7}{8}-\tfrac{5m-9}{3}\beta(m))L_{aa}f_{;m}\mathrm{Id}\}dy\,.
\end{aligned}$$

We shall give part of the proof Theorem 3.14.1 and refer to the discussion in [144, 193] in the interest of brevity.

The asymptotic coefficients $a_n(f,D_1,\mathcal{B}_s)$ are locally computable for $n < m$. The interior integrands vanish if n is odd while if $n = 0,2,4,6$, they are given by Theorem 3.3.1. The boundary integrands are homogeneous of weight $n-1$; Assertion (1) now follows.

Since the bundles V_1 and V_2 are distinct; we must use γ_m to identify V_1 and V_2 near the boundary. This observation reduces the number of invariants that are homogeneous of weight 1; for example, $\mathrm{Tr}\,(\psi)$ is not invariantly defined and thus does not appear in our list of invariants. After a bit of work, we see that there exist universal constants so that

$$a_1(f,D_1,\mathcal{B}_s)=b_1(m)(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\left\{f\mathrm{Id}\right\}dy,\qquad(3.14.a)$$

$$\begin{aligned}a_2(f,D_1,\mathcal{B}_s)&=(4\pi)^{-m/2}\tfrac{1}{6}\int_M\mathrm{Tr}\left\{f\tau\mathrm{Id}\,+6fE\right\}dx\\&+(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\left\{c_0(m)f(\hat\psi+\hat\psi^*)+c_1(m)f(\hat\psi-\hat\psi^*)\right.\\&\left.+c_2(m)f\Theta+c_3(m)fL_{aa}\mathrm{Id}\,+c_4(m)f_{;m}\mathrm{Id}\,\right\}dy\,.\end{aligned}$$

In contrast to the situation when the boundary operator $\mathcal{B}_s$ is local, the constants exhibit non-trivial dependence upon the dimension.

We begin our evaluation of the unknown coefficients with the following:

Lemma 3.14.2 $b_1(m)=\frac{1}{4}\left[\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac12)\Gamma(\frac{m+1}{2})}-1\right]$ *and* $c_0(m)=\frac12$.

Proof: We adopt the notation of Theorem 3.2.14. Let the structures be product near the boundary. This means that $P=\gamma_m(\nabla_{e_m}+B)$ where B is a self-adjoint tangential operator of Dirac type with coefficients that are independent of the normal variable. We assume that B is self-adjoint or equivalently that $\hat\psi=\hat\psi^*$. We take $\Theta=0$ so $A=B$.

Since B is defined on a manifold of dimension $m-1$, Theorem 3.13.1 implies that

$$a_1^\eta(B)=-(4\pi)^{-(m-1)/2}(m-2)\int_{\partial M}\mathrm{Tr}\,(\hat\psi)dy.\qquad(3.14.b)$$

We combine Theorem 3.2.14, Theorem 3.3.1, and Equation (3.14.b) to see

$$\begin{aligned}a_1(1,D_1,\mathcal{B}_s)&=\tfrac14\left(\tfrac{\Gamma(\frac{m}{2})}{\Gamma(\frac12)\Gamma(\frac{m+1}{2})}-1\right)(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\left\{\mathrm{Id}\right\}.\\a_2(1,D_1,\mathcal{B}_s)&=(4\pi)^{-m/2}\tfrac16\int_M\mathrm{Tr}\left\{f\tau\mathrm{Id}\,+6E\right\}dx\\&+\tfrac{1}{2(m-2)\Gamma(\frac12)}(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\left\{(m-2)\hat\psi\right\}dy\,.\end{aligned}$$

The desired identities now follow from the defining relations given in Equation (3.14.a) and by noting that $2\Gamma(\frac12)=(4\pi)^{1/2}$. $\square$

Next we study the coefficients $c_1(m)$ and $c_2(m)$.

Lemma 3.14.3 *We have* $c_1(m)=0$ *and* $c_2(m)=0$.

Proof: We apply Lemma 3.2.12 to see $c_2(m)=0$. To show that $c_1(m)=0$, we change the connection involved. Let

$$\nabla_{\varepsilon,e_i}:=\nabla_{e_i}-\sqrt{-1}\varepsilon h_i$$

where $H := h_i e_i$ is a smooth vector field on M. By Equation (1.1.k), ∇_ε is again a compatible unitary connection. One has

$$\begin{aligned}
\psi_\varepsilon &= \psi + \sqrt{-1}\varepsilon h_i \gamma_i, \\
\tilde\psi_\varepsilon &= \tilde\psi - \sqrt{-1}\varepsilon h_i \gamma_m \gamma_i, \\
A_\varepsilon &= A - \sqrt{-1}\varepsilon h_a \gamma_m \gamma_a, \\
\Theta_\varepsilon &= \Theta - \sqrt{-1}\varepsilon h_a \gamma_m \gamma_a, \\
\tilde\psi_\varepsilon + \tilde\psi_\varepsilon^* &= \tilde\psi + \tilde\psi^* - 2\varepsilon h_a \gamma_m \gamma_a, \\
\tilde\psi_\varepsilon - \tilde\psi_\varepsilon^* &= \tilde\psi - \tilde\psi^* + 2\sqrt{-1}\varepsilon h_m \mathrm{Id}\ .
\end{aligned}$$

Consequently as $\mathrm{Tr}\,(\gamma_m \gamma_a) = 0$ and as a_n is independent of the particular compatible connection which is chosen, one has

$$2\sqrt{-1} c_1(m)(4\pi)^{-m/2} \int_M \mathrm{Tr}\,(f h_m) dy = 0;$$

the desired conclusion $c_1(m) = 0$ now follows. □

The remaining coefficients $b_1(m)$, $c_3(m)$, and $c_4(m)$ are determined by making a special case computation on the ball. We shall omit the details and instead refer to Dowker et. al. [144] for details as it is a bit outside the line of development that we have been following. We also refer to Kirsten [258] for an excellent discussion of the method of special case computations.

3.15 Non-minimal operators

We follow the discussion in [83] for the most part and refer to related work in [22, 239, 293]. Let A and B be positive constants. As in Section 2.11, we consider an operator of the form

$$D_E^p := A d\delta + B\delta d - E \quad \text{on} \quad C^\infty(\Lambda^p(M))\,.$$

If ∂M is non-empty, we let $\mathcal{B}$ define either absolute or relative boundary conditions. Then the analysis of Section 1.5.4 extends to show that $(D_E^p, \mathcal{B})$ is elliptic with respect to the cone $\mathcal{C}$.

We begin by treating the case $E = 0$.

Theorem 3.15.1 *Let $D^p := A d\delta + B\delta d$ on $C^\infty(\Lambda^p(M))$ where M is a compact Riemannian manifold with smooth boundary. Let $\mathcal{B}$ define either absolute or relative boundary conditions. Then:*

$$\begin{aligned}
a_n(1, D^p, \mathcal{B}) &= B^{(n-m)/2} a_n(1, \Delta^p, \mathcal{B}) \\
&+ (B^{(n-m)/2} - A^{(n-m)/2}) \textstyle\sum_{k<p} (-1)^{k+p} a_n(1, \Delta^p, \mathcal{B})\,.
\end{aligned}$$

If $E \neq 0$, then the situation is considerably more complicated. Define

$$c_{m,p}(A, B) := B^{-m} + (B^{-m} - A^{-m}) \sum_{k<p} (-1)^{k+p} \binom{m}{p}^{-1} \binom{m}{k}\,. \qquad (3.15.a)$$

Theorem 3.15.2 *Let $D_E^p := Ad\delta + B\delta d - E$ on the bundle of smooth p forms where M is a compact Riemannian manifold M with smooth boundary. Let $\mathcal{B}$ define either absolute or relative boundary conditions. Then:*

1. $a_0(1, D_E^p, \mathcal{B}) = a_0(1, D^p, \mathcal{B})$.
2. $a_1(1, D_E^p, \mathcal{B}) = a_1(1, D^p, \mathcal{B})$.
3. $a_2(1, D_E^p, \mathcal{B}) = a_2(1, D^p, \mathcal{B}) + (4\pi)^{-m/2} c_{m,p}(A, B) \int_M \operatorname{Tr}(E)dx$.

Proof of Theorem 3.15.1 We apply an argument similar to that used to establish Lemma 2.1.16. Let $E_{\lambda,\mathcal{B}}^p$ be the eigenspaces of the realization of the Laplacian defined by the boundary conditions $\mathcal{B}$. By Lemma 1.5.6, we have two long exact sequences

$$\begin{aligned}
&0 \to E_{\lambda,\mathcal{B}}^0 \xrightarrow{d} E_{\lambda,\mathcal{B}}^1 \xrightarrow{d} \ldots \xrightarrow{d} E_{\lambda,\mathcal{B}}^{m-1} \xrightarrow{d} E_{\lambda,\mathcal{B}}^m \to 0 \\
&0 \leftarrow E_{\lambda,\mathcal{B}}^0 \xleftarrow{\delta} E_{\lambda,\mathcal{B}}^1 \xleftarrow{\delta} \ldots \xleftarrow{\delta} E_{\lambda,\mathcal{B}}^{m-1} \xleftarrow{\delta} E_{\lambda,\mathcal{B}}^m \leftarrow 0\,.
\end{aligned}$$

If $\lambda \neq 0$, then we may decompose

$$E_{\lambda,\mathcal{B}}^p = \{\text{image}\,(d^{p-1}) \cap E_{\lambda,\mathcal{B}}^p\} \oplus \{\text{image}\,(\delta^p) \cap E_{\lambda,\mathcal{B}}^p\}\,.$$

We define

$$\begin{aligned}
&f_p(t, d, \mathcal{B}) := \sum_{\lambda>0} e^{-t\lambda} \dim \left\{\text{image}\,(d^{p-1}) \cap E_{\lambda,\mathcal{B}}^p\right\}, \\
&f_p(t, \delta, \mathcal{B}) := \sum_{\lambda>0} e^{-t\lambda} \dim \left\{\text{image}\,(\delta^p) \cap E_{\lambda,\mathcal{B}}^p\right\}, \\
&\beta_p := \ker(\Delta_{\mathcal{B}}^p)\,.
\end{aligned}$$

Set $f_{-1}(t, \delta, \mathcal{B}) = 0$ and $f_0(t, d, \mathcal{B}) = 0$. We may then express

$$\begin{aligned}
&\operatorname{Tr}_{L^2}(e^{-t\Delta_{\mathcal{B}}^p}) = f_p(t, d, \mathcal{B}) + f_p(t, \delta, \mathcal{B}) + \beta_p, \quad \text{and} \\
&\operatorname{Tr}_{L^2}(e^{-tD_{p,\mathcal{B}}}) = f_p(At, d, \mathcal{B}) + f_p(Bt, \delta, \mathcal{B}) + \beta_p\,.
\end{aligned} \tag{3.15.b}$$

On the other hand, as d^{p-1} is an isomorphism from image $(\delta^p) \cap E_{\lambda,\mathcal{B}}^p$ to image $(d^p) \cap E_{\lambda,\mathcal{B}}^{p+1}$ for $\lambda \neq 0$, we have that

$$f_p(t, d, \mathcal{B}) = f_{p-1}(t, \delta, \mathcal{B})\,. \tag{3.15.c}$$

Consequently, we have the shuffle formula

$$\begin{aligned}
f_p(t, \delta, \mathcal{B}) &= \operatorname{Tr}_{L^2}(e^{-t\Delta_{\mathcal{B}}^p}) - \beta_p - f_p(t, d, \mathcal{B}) \\
&= \operatorname{Tr}_{L^2}(e^{-t\Delta_{\mathcal{B}}^p}) - \beta_p - f_{p-1}(t, \delta, \mathcal{B})
\end{aligned}$$

which then leads by induction to the formula

$$f_p(t, \delta, \mathcal{B}) = \sum_{k=0}^{p} (-1)^{k+p} \left\{\operatorname{Tr}_{L^2}(e^{-t\Delta_{\mathcal{B}}^k}) - \beta_k\right\}. \tag{3.15.d}$$

We combine Equations (3.15.c) and (3.15.d) to see

$$f_p(t,d,\mathcal{B}) = \sum_{k=0}^{p-1}(-1)^{k+p-1}\left\{\mathrm{Tr}_{L^2}(e^{-t\Delta_{\mathcal{B}}^k}) - \beta_k\right\}. \tag{3.15.e}$$

We now apply Equations (3.15.b), (3.15.d), and (3.15.e) to conclude

$$\begin{aligned}\mathrm{Tr}_{L^2}\{e^{-tD_{\mathcal{B}}^p}\} &= \beta_p + \sum_{k=0}^{p}(-1)^{k+p}\left\{\mathrm{Tr}_{L^2}(e^{-Bt\Delta_{\mathcal{B}}^k}) - \beta_k\right\} \\ &- \sum_{k=0}^{p-1}\left\{\mathrm{Tr}_{L^2}(e^{-At\Delta_{\mathcal{B}}^k}) - \beta_k\right\}.\end{aligned}$$

The Betti numbers cancel and we are left with the expression

$$\begin{aligned}&\mathrm{Tr}_{L^2}(e^{-tD_{\mathcal{B}}^p}) = \mathrm{Tr}_{L^2}(e^{-Bt\Delta_{\mathcal{B}}^p}) \\ &\quad + \sum_{k<p}(-1)^{k+p}\left\{\mathrm{Tr}_{L^2}(e^{-Bt\Delta_{\mathcal{B}}^k}) - \mathrm{Tr}_{L^2}(e^{-At\Delta_{\mathcal{B}}^k})\right\}.\end{aligned}$$

We equate coefficients in the asymptotic expansions to complete the proof of Theorem 3.15.1. □

Proof of Theorem 3.15.2 Assertions (1) and (2) follow since E has weight 2 and thus does not enter into the invariants a_0 and a_1. Furthermore, the boundary integrand for a_2 does not involve E. Thus we may assume without loss of generality that M is a closed manifold in the proof of Assertion (3).

We must recall a bit of the construction of the invariants a_n using the Seeley calculus [339, 340]. Let P be a second order differential operator on a smooth vector bundle V over a closed Riemannian manifold M. If we fix a local frame for V and choose local coordinates on M, then we may express

$$P = p_2^{ij}(x)\partial_i\partial_j + p_1^k(x)\partial_k + p_0(x).$$

The *total symbol* of P is given by

$$\begin{aligned}&\sigma(P)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x) \quad \text{where} \\ &p_2(x,\xi) := -p_2^{ij}(x)\xi_i\xi_j, \quad \text{and} \quad p_1(x,\xi) := \sqrt{-1}p^i(x)\xi_i.\end{aligned}$$

We suppose P is self-adjoint and elliptic with respect to the cone $\mathcal{C}$. This means that $p_2(x,\xi)$ is self-adjoint and that the eigenvalues are positive for $\xi \neq 0$. Let γ be a curve around $[0,\infty)$ in $\mathbb{C}$. If $\lambda \in \gamma$, define

$$r_0(x,\xi,\lambda) := (p_2(x,\xi) - \lambda)^{-1}$$

and inductively for $n > 0$

$$r_n(x,\xi,\lambda) := -\sum_{n=j+|\alpha|+2-k,\, j<n} \tfrac{1}{\sqrt{-1}^\alpha \alpha!}\partial_\xi^\alpha r_j \cdot \partial_x^\alpha p_k \cdot r_0$$

where $\alpha = (\alpha_1, ..., \alpha_m)$ is a multi index. There is a universal constant c_m so

$$a_n(1,P) = c_m \int e^{-\lambda} \mathrm{Tr}\left\{ r_n(x,\xi,\lambda) \right\} d\lambda d\xi dx \,.$$

We specialize to the case $P = Ad\delta + B\delta d - E$ on $C^\infty(\Lambda^p(M))$. Expand

$$r_2(x,\xi,\lambda) = -r_0 E r_0$$

modulo terms which are linear in the second derivatives of the metric and quadratic in the first derivatives of the metric. We suppress these terms as we are interested only in the dependence upon E in the formula to compute

$$\begin{aligned} a_2(1, D^p_E, \mathcal{B}) &= -c_m \int e^{-\lambda} \mathrm{Tr}\left\{ r_0 E r_0 \right\} d\lambda d\xi dx + ... \\ &= -c_m \int \mathrm{Tr}\left\{ E \int e^{-\lambda} r_0^2 d\lambda d\xi \right\} dx + ... \end{aligned}$$

We use Lemma 1.2.5 to see

$$p_2(x,\xi) = A\mathfrak{e}(\xi)\mathfrak{i}(\xi) + B\mathfrak{i}(\xi)\mathfrak{e}(\xi) \,.$$

Consequently

$$\Psi_{m,p}(A,B) := \int e^{-\lambda} r_0^2 d\lambda d\xi$$

is a universally defined natural endomorphism which is self-adjoint and which depends smoothly on the parameters A and B. Since the action of the orthogonal group on Λ^p is irreducible, $\Psi_{A,B}$ is a universal multiple of the identity. Consequently, there exist universal constants $c_{m,p}(A,B)$ so that

$$a_2(1, D^p_E, \mathcal{B}) = a_2(1, D^p, \mathcal{B}) + \int_M c_{m,p}(A,B) \mathrm{Tr}\,\{E\} dx \,.$$

The universal constants $c_{m,p}(A,B)$ can be evaluated by setting $E = 1$. Lemma 3.1.1 generalizes immediately to this setting. We have

$$\mathrm{Tr}_{L^2}\{e^{-t(D^p_{\mathcal{B}} - \mathrm{Id})}\} = e^t \mathrm{Tr}_{L^2}\{e^{-tD^p_{\mathcal{B}}}\}$$

so

$$a_2(1, D^p - \mathrm{Id}, \mathcal{B}) = a_2(1, D^p, \mathcal{B}) + a_0(1, D^p, \mathcal{B}) \,.$$

Since $a_0(1, \Delta^k, \mathcal{B}) = \binom{m}{k} \int_M dx$, the universal constant $c_{m,p}(A,B)$ can now be evaluated from Assertion (1) and is seen to be given by the expression in Equation (3.15.a). $\square$

Remark 3.15.3 The analysis of Avramidi and Branson [22] shows that in general the formulae for non-minimal operators is extraordinarily complicated. The present setting is relatively easy to treat owing to the naturality of the operators involved and the shuffle formulae that intertwine the relevant operators.

3.16 Fourth order operators

Let M be a closed Riemannian manifold. The scalar Laplacian is given by $\Delta f = -f_{;ii}$. There are several natural fourth order operators on $C^\infty(M)$ whose leading symbol is given by the square of the metric tensor. For example, one could define

$$P_1 f := f_{;iijj}, \quad P_2 f := f_{;ijji}, \quad \text{and} \quad P_3 f := f_{;ijij}\,. \tag{3.16.a}$$

The *symmetrized* 4^{th} *order scalar Laplacian* is the average of these operators

$$\tilde{\Delta}_2 f: \;=\; \tfrac{1}{3}\{f_{;iijj} + f_{;ijij} + f_{;ijji}\} = f_{;iijj} + \tfrac{2}{3}\rho_{jk} f_{;jk} - \tfrac{1}{3}\tau_{;k} f_{;k};$$

we refer to Gray and Willmore [222] for details concerning this and other related operators.

Let ∇ be a connection on a vector bundle V over a closed Riemannian manifold M. We consider a 4^{th} order operator of the form:

$$Pu = u_{;iijj} + p_{2,ij} u_{;ij} + p_{1,i} u_{;i} + p_0 \tag{3.16.b}$$

where $p_{2,ij} = p_{2,ji}$ and where $\{p_{2,ij}, p_{1,i}, p_0\}$ are endomorphism valued. The 3 operators given in Display (3.16.a) can all be put in this form as can be the symmetrized 4^{th} order scalar Laplacian. As noted in Section 3.2, the function $\Gamma(\frac{m-s}{d})^{-1}\Gamma(\frac{m-s}{kd})$ is entire. Set

$$\Gamma(\tfrac{m-n}{d})^{-1}\Gamma(\tfrac{m-n}{kd}) := \lim_{s\to n}\{\Gamma(\tfrac{m-s}{d})^{-1}\Gamma(\tfrac{m-s}{kd})\}\,.$$

We assume M closed in the interest of simplicity. We shall also suppress the divergence terms to study the invariants $a_n(1,P)$.

Theorem 3.16.1 *Let M be a closed Riemannian manifold of dimension m. Let P be as given in Equation* (3.16.b). *Then:*

1. $a_0(1,P) = \frac{1}{2}(4\pi)^{-m/2}\Gamma(\frac{m}{2})^{-1}\Gamma(\frac{m}{4})\int_M \operatorname{Tr}(I)dx.$
2. $a_2(1,P) = \frac{1}{2}(4\pi)^{-m/2}\Gamma(\frac{m-2}{2})^{-1}\Gamma(\frac{m-2}{4})\frac{1}{6}\int_M \operatorname{Tr}\{\tau\mathrm{Id} + \frac{3}{m}p_{2,ii}\}dx.$
3. $a_4(1,P) = \frac{1}{2}(4\pi)^{-m/2}\Gamma(\frac{m}{2})^{-1}\Gamma(\frac{m}{4})\frac{1}{360}\int_M \operatorname{Tr}\{\frac{90}{m+2}p_{2,ij}p_{2,ij} + \frac{45}{m+2}p_{2,ii}p_{2,jj}$
   $+(m-2)(5\tau^2\mathrm{Id} - 2|\rho|^2\mathrm{Id} + 2|R|^2\mathrm{Id} + 30\Omega_{ij}\Omega_{ij}) + 30\tau p_{2,ii}$
   $-60\rho_{ij}p_{2,ij} - 360p_0\}dx.$

Let P be given by Equation (3.16.b). We suppress divergence terms. We introduce the non-zero normalizing constants

$$\tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4}) \quad \text{and} \quad \tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m-2}{2})^{-1}\Gamma(\tfrac{m-2}{4})$$

and use dimensional analysis to see there are universal constants so that

$$\begin{aligned} a_0(1,P) &= \tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4})c_{0,m}\int_M \operatorname{Tr}\Big\{I\Big\}dx, \qquad (3.16.c)\\ a_2(1,P) &= \tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m-2}{2})^{-1}\Gamma(\tfrac{m-2}{4})\tfrac{1}{6}\int_M \operatorname{Tr}\Big\{c_{1,m}\tau\mathrm{Id} \end{aligned}$$

$$
\begin{aligned}
&+ \quad c_{2,m}p_{2,ii}\Big\}dx,\\
a_4(1,P) &= \tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4})\tfrac{1}{360}\int_M \mathrm{Tr}\Big\{c_{3,m}\tau^2\mathrm{Id} + c_{4,m}|\rho|^2\mathrm{Id}\\
&+ \quad c_{5,m}|R|^2\mathrm{Id} + c_{6,m}\Omega_{ij}\Omega_{ij} + c_{7,m}\tau p_{2,ii} + c_{8,m}\rho_{ij}p_{2,ij}\\
&+ \quad c_{9,m}p_{2,ii}p_{2,jj} + c_{10,m}p_{2,ij}p_{2,ij} + c_{11,m}p_0\Big\}dx\,.
\end{aligned}
$$

The non-commutativity of the endomorphisms in the vector bundle case plays no role and consequently we may restrict to the scalar case and omit "Tr" for the most part henceforth. If P_i are operators with leading symbols given by $|\xi_i|^4\mathrm{Id}$ on manifolds M_i, then the leading symbol of $P_1 + P_2$ is not given by $\{|\xi_1|^2 + |\xi_2|^2\}^2\mathrm{Id}$ and hence is no longer an operator in our category. Thus product formulae do not imply that the coefficients are dimension free. As in the case of spectral boundary conditions, the coefficients involve non-trivial dependence upon the dimension of the underlying manifold as is shown by Theorem 3.16.1.

We begin the proof of Theorem 3.16.1 by showing

Lemma 3.16.2 *1.* $c_{0,m} = 1$*,* $c_{1,m} = 1$*, and* $c_{2,m} = \frac{3}{m}$*.*

2. $c_{3,m} = 5(m-2)$*,* $c_{4,m} = -2(m-2)$*,* $c_{5,m} = 2(m-2)$*, and* $c_{6,m} = 30(m-2)$*.*

3. $2mc_{7,m} + 2c_{8,m} = 60(m-2)$*.*

4. $4m^2c_{9,m} + 4mc_{10,m} + c_{11,m} = 180(m-2)$*.*

Proof: Let ε be a real parameter. Let $Df := -(f_{;ii} + \varepsilon f)$. Expand

$$D^2 f = f_{;iijj} + 2\varepsilon f_{;ii} + \varepsilon^2 f\,.$$

Thus $p_{2,ij}(\varepsilon) = 2\varepsilon\delta_{ij}$ and $p_0 = \varepsilon^2$. By Lemma 3.2.17,

$$a_n(1,D^2) = \tfrac{1}{2}\Gamma(\tfrac{m-n}{2})^{-1}\Gamma(\tfrac{m-n}{4})a_n(1,D)\,. \tag{3.16.d}$$

We use Theorem 3.3.1 to compute $a_n(1,D)$ for $n = 0,2,4$. Consequently after taking into account the normalizing constants in Display (3.16.c), we have

$$
\begin{aligned}
&c_{0,m}\int_M dx = \int_M dx,\\
&\int_M (c_{1,m}\tau + 2c_{2,m}\varepsilon\delta_{ii})dx = \int_M (\tau + 6\varepsilon)dx\,.
\end{aligned}
$$

The first assertion of the Lemma now follows.

We use the functional relation $s\Gamma(s) = \Gamma(s+1)$ to express

$$
\begin{aligned}
&\Gamma(\tfrac{m-4}{2})^{-1}\Gamma(\tfrac{m-4}{4}) = \tfrac{m-4}{2}\tfrac{m-2}{2}\tfrac{4}{m-4}\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4}) \\
&= \quad (m-2)\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4})\,.
\end{aligned} \tag{3.16.e}
$$

Thus Equation (3.16.d) and Theorem 3.3.1 yield

$$\int_M \Big\{c_{3,m}\tau^2 + c_{4,m}|\rho|^2 + c_{5,m}|R|^2 + c_{6,m}\Omega_{ij}\Omega_{ij}$$

$$
\begin{aligned}
&+\quad (2mc_{7,m}+2c_{8,m})\tau\varepsilon+(4m^2c_{9,m}+4mc_{10,m}+c_{11,m})\varepsilon^2\Big\}dx\\
&=\quad (m-2)\int_M\Big\{5\tau^2-2|\rho|^2+2|R|^2+60\tau\varepsilon+30\Omega_{ij}\Omega_{ij}+180\varepsilon^2\Big\}dx\,.
\end{aligned}
$$

We equate coefficients to complete the proof. □

We complete the proof of Theorem 3.16.1 by establishing:

Lemma 3.16.3

1. $c_{11,m}=-360$.
2. $c_{9,m}=\frac{45}{m+2}$, *and* $c_{10,m}=\frac{90}{m+2}$.
3. $c_{7,m}=30$, $c_{8,m}=-60$.

Proof: By Lemma 3.2.16,

$$a_4(F,P-\varepsilon)=a_4(F,P)+\varepsilon a_0(F,P)\,.$$

Because $p_0=-\varepsilon$, $c_{11,m}=-360c_{0,m}$. Consequently, Assertion (1) follows from Lemma 3.16.2.

By Lemma 3.16.2 (2) and Assertion (1),

$$mc_{9,m}+c_{10,m}=45\,. \tag{3.16.f}$$

To complete the proof of Assertion (2), we derive another relationship between $c_{9,m}$ and $c_{10,m}$. Give the flat dimensional torus $M=\mathbb{T}^m$ the product metric. Let Ξ_{ij} be a collection of self-adjoint *Pauli matrices* which satisfy the relations:

$$\Xi_{ij}=\Xi_{ji}\quad\text{and}\quad\Xi_{ij}\Xi_{kl}+\Xi_{kl}\Xi_{ij}=\{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\}\,. \tag{3.16.g}$$

Let $Q_\varepsilon f:=\Xi_{ij}f_{;ij}+\varepsilon f$ and let $P_\varepsilon:=Q_\varepsilon^2$. Then

$$
\begin{aligned}
P_\varepsilon f &= \Xi_{ij}\Xi_{kl}f_{;ijkl}+2\varepsilon\Xi_{ij}f_{;ij}+\varepsilon^2 f\\
&= \tfrac12\{\Xi_{ij}\Xi_{kl}+\Xi_{kl}\Xi_{ij}\}f_{;ijkl}+2\varepsilon\Xi_{ij}f_{;ij}+\varepsilon^2 f\\
&= f_{;iijj}+2\varepsilon\Xi_{ij}f_{;ij}+\varepsilon^2 f\,.
\end{aligned}
$$

By Lemma 3.2.18 with $n=d=4$, $\partial_\varepsilon^2 a_4(1,P_\varepsilon)=(m-2)a_0(1,P_0)$ so

$$
\begin{aligned}
&\tfrac{2}{360}\int_M\operatorname{Tr}\Big\{4c_{9,m}\Xi_{ii}\Xi_{jj}+4c_{10,m}\Xi_{ij}\Xi_{ij}+c_{11,m}\Big\}dx\\
=\quad &(m-2)\int_M\operatorname{Tr}\Big\{\operatorname{Id}\Big\}dx\,.
\end{aligned}
$$

The commutation relations of Equation (3.16.g) show that

$$
\begin{aligned}
\operatorname{Tr}\{\Xi_{ii}\Xi_{jj}\} &= \operatorname{Tr}(\operatorname{Id})\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\ne j,\end{cases}\\
\operatorname{Tr}\{\Xi_{ij}\Xi_{ij}\} &= \operatorname{Tr}(\operatorname{Id})\begin{cases}1 & \text{if } i=j,\\ \frac12 & \text{if } i\ne j\,.\end{cases}
\end{aligned}
$$

This leads to the relationship $4mc_{9,m} + 2m(m+1)c_{10,m} - 360 = 180(m-2)$ or equivalently that

$$2c_{9,m} + (m+1)c_{10,m} = 90\,. \tag{3.16.h}$$

We solve Equations (3.16.f) and (3.16.h) for $c_{9,m}$ and $c_{10,m}$ to establish the validity of Assertion (2).

By Lemma 3.16.2 (3),

$$2mc_{7,m} + 2c_{8,m} = 60(m-2)\,. \tag{3.16.i}$$

Suppose that $m = 2$. By Equation (3.16.i), $4c_{7,2} + 2c_{8,2} = 0$. As $\rho_{jk} = \frac{1}{2}\delta_{jk}\tau$,

$$\int_M \left\{ c_{7,2}\tau p_{2,ii} + c_{8,2}\rho_{ij}p_{2,ij} \right\} dx = (c_{7,m} + \tfrac{1}{2}c_{8,2}) \int_M \left\{ \tau p_{2,ii} \right\} dx = 0\,.$$

Thus we may take $c_{7,2} = 30$ and $c_{8,2} = -60$. We therefore suppose $m > 2$. One relationship between the coefficients $c_{7,m}$ and $c_{8,m}$ is provided by Equation (3.16.i). We shall derive an independent second relationship between $c_{7,m}$ and $c_{8,m}$ to complete the proof of the Lemma.

Let $V_1 = V_2 = M \times \mathbb{C}$ be the trivial line bundle over a closed m dimensional Riemannian manifold M and let Θ be a smooth co-vector field on M. Give these bundles the flat connection. We define

$$\begin{aligned} A_\Theta f : &= f_{;ii} + 2\Theta_i f_{;i} + \Theta_{i;i} f & &: C^\infty(V_1) \to C^\infty(V_2), \\ A_\Theta^* f : &= f_{;ii} - 2\Theta_i f_{;i} - \Theta_{i;i} f = A_{-\Theta} f & &: C^\infty(V_2) \to C^\infty(V_1)\,. \end{aligned}$$

The index of this elliptic complex vanishes if $\Theta = 0$ and hence vanishes for all Θ as the index is a continuous integer valued invariant. Therefore by Theorem 1.3.9,

$$a_n(1, A^*A) - a_n(1, AA^*) = 0 \quad \text{for all} \quad n\,.$$

We use Equation (1.1.g) to commute covariant derivatives and to see that

$$2\Theta_i f_{;ijj} - 2\Theta_j f_{;iij} = 2\Theta_i f_{;jij} - 2\Theta_j f_{;iji} + 2\rho_{ik}\Theta_i f_{;k} = 2\rho_{ik}\Theta_i f_{;k}\,.$$

We may then compute

$$\begin{aligned} A^*Af &= f_{;iijj} + 2(\Theta_{j;k} + \Theta_{k;j})f_{;jk} \\ &+ 2(\Theta_{k;jj} + \Theta_{j;jk} + \rho_{jk}\Theta_j)f_{;k} + \Theta_{j;jkk} f + O(\Theta^2), \\ AA^*f &= f_{;iijj} - 2(\Theta_{j;k} + \Theta_{k;j})f_{;jk} \\ &- 2(\Theta_{k;jj} + \Theta_{j;jk} + \rho_{jk}\Theta_j)f_{;k} - \Theta_{j;jkk} f + O(\Theta^2)\,. \end{aligned}$$

The vanishing of a_4 then leads to the relation

$$\begin{aligned} 0 &= \int_M \left\{ 8c_{7,m}\tau\Theta_{k;k} + 8c_{8,m}\rho_{jk}\Theta_{j;k} \right\} dx \\ &= -\int_M \left\{ 8c_{7,m}\tau_{;k}\Theta_k + 8c_{8,m}\rho_{jk;k}\Theta_j \right\} dx \\ &= -\int_M \left\{ 8c_{7,m} + 4c_{8,m} \right\} \tau_{;k}\Theta_k dx\,. \end{aligned}$$

Setting $\Theta = d\tau$ then implies $2c_{7,m} + c_{8,m} = 0$. We combine this relation with Equation (3.16.i) to complete the proof of the Lemma and thereby also of Theorem 3.16.1. $\square$

We have restricted to 4^{th} order operators in the interest of simplicity; similar methods can be used to study higher order operators. We omit details in the interest of brevity and refer instead to [180] for further details.

3.17 Pseudo-differential operators

We begin by reviewing some facts which we shall need concerning classic pseudo-differential operators; we refer to Agranovič [1], to Duistermaat and Guillemin [152], to Greiner [224], to Grubb [229], to Grubb and Seeley [234], and to Seeley [339, 341] for further details.

If f is a function with compact support in an open subset $\mathcal{O}$ of $\mathbb{R}^m$, then the *Fourier transform* of f is defined by

$$\hat{f}(\xi) := (2\pi)^{-m/2} \int e^{-\sqrt{-1}x\cdot\xi} f(x)dx\,.$$

The *Fourier inversion formula* expresses f dually as

$$f(x) = (2\pi)^{-m/2} \int e^{\sqrt{-1}x\cdot\xi} \hat{f}(\xi)d\xi\,.$$

Let $P = \sum_{\vec{a}} p_{\vec{a}}(x)\partial_{\vec{a}}^{x}$ be a partial differential operator and let p be the *total symbol* of P;

$$p(x,\xi) := \sum_{\vec{a}} p_{\vec{a}}(x)(\sqrt{-1})^{|\vec{a}|}\xi^{\vec{a}}\,.$$

We may then use the Fourier inversion formula to see that

$$\begin{aligned} Pf(x) &= (2\pi)^{-m/2} \sum_{\vec{a}} p_{\vec{a}}(x)\partial_x^{\vec{a}} \int e^{\sqrt{-1}x\cdot\xi}\hat{f}(\xi)d\xi \\ &= (2\pi)^{-m/2} \int e^{\sqrt{-1}x\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi\,. \end{aligned}$$

More generally, given a suitable symbol $p(x,\xi)$, we can define an operator $P = P_p$ on $C_0^\infty(\mathcal{O})$ by setting

$$Pf(x) = (2\pi)^{-m/2} \int e^{\sqrt{-1}x\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi\,.$$

We shall suppose p has compact x support in $\mathcal{O}$, that p is smooth in (x,ξ), and that p satisfies estimates of the form

$$|d_x^{\vec{a}} d_\xi^{\vec{b}} p(x,\xi)| \le C_{\vec{a},\vec{b}}(1+|\xi|)^{d-|\beta|}\,.$$

We also assume there exist symbols $p_{d-j}(x,\xi)$ with $p_{d-j}(x,t\xi) = t^{d-j}(x,\xi)$

for $t \geq 1$ and $|\xi| \geq 1$ so that the difference $p - p_d - ... - p_{d-j}$ satisfies similar estimates for any j where d is replaced by $d - j - 1$. In this situation, we say that P is a *classic pseudo-differential operator*. The passage to the context operators on vector bundles over closed manifolds is relatively straightforward using suitable partitions of unity.

Let $d \in \mathbb{N} = \{1, 2, ...\}$ be a positive integer henceforth. We say that a classic pseudo-differential operator P of order d on $C^\infty(V)$ is elliptic with respect to the cone $\mathcal{C}$ if $p_d - \lambda$ is invertible for $|\xi| \geq 1$ and $\lambda \in \mathcal{C}$. Let $\Psi_d(M, V)$ be the space of all such operators and let $P_d(M, V) \subset \Psi_d(M, V)$ be the subspace of partial differential operators. We shall work in the self-adjoint setting for technical convenience only; most of the results given subsequently are valid, with suitable modifications, in the setting of classic pseudo-differential operators which are elliptic with respect to the cone $\mathcal{C}$.

Let ∇ be a unitary connection on a Hermitian vector bundle V. We construct a self-adjoint positive second order partial differential operator which is elliptic with respect to the cone $\mathcal{C}$ by setting

$$D_2 u := -u_{;ii} + 1\,. \tag{3.17.a}$$

This operator will play a central role in what follows. Since $D_2^k \in P_{2k}(M, V)$, these spaces are non-empty for any $k \in \mathbb{N}$. It will follow from the following result, due to Seeley, that $D_2^{d/2} \in \Psi_d(M, V)$ and hence these spaces are non-empty as well.

Lemma 3.17.1 *Let M be a closed manifold. If $P \in \Psi_d(M, V)$, then P^s is a self-adjoint classic pseudo-differential operator for any $s \in \mathbb{C}$.*

A new feature of the heat trace asymptotics for classic pseudo-differential operators which is not present for differential operators is the presence of log singularities. This is caused by the fact that the symbols in question must be smoothed off near $\xi = 0$ and are not homogeneous for $|\xi| < 1$.

Theorem 3.17.2 *Let M be a closed manifold. If $P \in \Psi_d(M, V)$, then the fundamental solution of the heat equation, e^{-tP}, is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion*

$$\operatorname{Tr}_{L^2}\{e^{-tP}\} \sim \sum_{n=0}^{\infty} a_n(P) t^{(n-m)/d} + \sum_{k=1}^{\infty} b_k(P) t^k \log(t)\,.$$

The invariants $b_k(P)$ are locally computable for all $k \in \mathbb{N}$. If $\frac{n-m}{d} \notin \mathbb{N}$, then the invariants $a_n(P)$ are locally computable as well.

Remark 3.17.3 One can also introduce a smearing endomorphism F or even more generally consider $\operatorname{Tr}_{L^2}\{Qe^{-tP}\}$, where Q is a suitable chosen auxiliary operator, to define heat trace invariants. We omit details in the interest of brevity.

We have a corresponding ζ function expression of these invariants. The analysis of Section 1.3.4 extends to yield:

Theorem 3.17.4 *Let M be a closed manifold. If $P \in \Psi_d(M,V)$, then the zeta function $\zeta(s,P) := \mathrm{Tr}_{L^2}\{P^{-s}\}$ is well defined for $\Re(s) > \frac{m}{d}$. It has a meromorphic extension to $\mathbb{C}$ with isolated simple poles at $s = (m-n)/d$ and $n = 0,1,2,\ldots$. We have*

$$\begin{aligned} a_n(P) &= \mathrm{res}_{\,s=(m-n)/d}\{\Gamma(s)\zeta(s,P)\}, \\ b_k(P) &= -\mathrm{res}_{\,s=-k}\{(s+k)\Gamma(s)\zeta(s,P)\}\,. \end{aligned}$$

We say that a property holds *generically* on $\Psi_d(M,V)$ or on $P_d(M,V)$ if the corresponding set of symbols defining operators for which the property holds is a Baire category II subset in the appropriate C^∞ topology.

If a d^{th} order partial differential operator is elliptic with respect to the cone $\mathcal{C}$, then necessarily d is even. In this setting, the invariants $b_k(P)$ vanish for all k. Furthermore $a_n(P) = 0$ for n odd. The following result was first established by Gilkey and Grubb [192] and we shall follow the development there for the most part. We also refer to earlier related work by Cognola et. al. [125].

Theorem 3.17.5 *Let M be a closed manifold. Let $P \in \Psi_d(M,V)$.*

1. *If $a_i(P) \neq 0$ for $0 \leq i < d$, then $a_n(P + \varepsilon\mathrm{Id}\,) \neq 0$ for all n for generic values of $\varepsilon > 0$.*
2. *If $b_1(P) \neq 0$, then $b_k(P + \varepsilon\mathrm{Id}\,) \neq 0$ for all k for generic values of $\varepsilon > 0$.*
3. *The invariants $\{a_0(\cdot),\ldots,a_{d-1}(\cdot)\}$ are generically non-zero on $\Psi_d(M,V)$.*
4. *If d is even, then $a_{2n}(\cdot)$ is generically non-zero on $P_d(M,V)$.*
5. *The invariant $b_1(\cdot)$ is generically non-zero on $\Psi_d(M,V)$.*
6. *If $(n-m)/d \in \mathbb{N}$, then $a_n(\cdot)$ is not locally computable on $\Psi_d(M,V)$.*

The remainder of this section is devoted to the proof of Theorem 3.17.5. In the proof of Assertions (1)-(4), we introduce a sequence of parameters ε_i and ϱ_j; these are assumed to be small positive real numbers.

Let $P \in \Psi_d(M,V)$. Set $P_\varepsilon := P + \varepsilon\mathrm{Id}$. As $\mathrm{Tr}_{L^2}\{e^{-tP_\varepsilon}\} = e^{-t\varepsilon}\mathrm{Tr}_{L^2}\{e^{-tP}\}$,

$$a_n(P_\varepsilon) = \sum_{j+dk=n} \tfrac{1}{k!}(-\varepsilon)^k a_j(P) \quad \text{and} \quad b_\ell(P_\varepsilon) = \sum_{j+k=\ell} \tfrac{1}{k!}(-\varepsilon)^k b_j(P)\,.$$

The invariants $a_n(P_\varepsilon)$ are polynomial in ε. If $a_i(P) \neq 0$ for $0 \leq i < d$, then these polynomials do not vanish identically and hence for fixed n have only a finite number of zeros. Assertion (1) now follows; the proof of Assertion (2) is similar.

The invariants $\{a_0(\cdot),\ldots,a_{d-1}(\cdot),b_1(\cdot)\}$ are given by local formulae. Thus the set of all P where such an invariant is non-zero is an open subset of $\Psi_d(M,V)$. To prove Assertions (3) and (4), it suffices to show that the set where such an invariant is non-zero is dense in $\Psi_d(M,V)$.

The Seeley calculus yields

$$a_0(P) = (2\pi)^{-m}\frac{1}{2\pi\sqrt{-1}}\int \mathrm{Tr}\left\{\int e^{-\lambda}(p_d(x,\xi)-\lambda)^{-1}d\lambda\right\}d\xi dx$$

where the $d\lambda$ integral runs over a suitably chosen contour in the complex plane, where the $d\xi$ integral runs over the fiber of the cotangent bundle over x, and where the dx integral ranges over M.

The Cauchy integral formula shows that the $d\lambda$ integral defines an operator whose eigenvalues are the negative exponential of the eigenvalues of p_d. As these eigenvalues are necessarily positive, the trace is positive and exponentially decreasing in ξ. Thus the $d\xi dx$ integral is absolutely convergent so $a_0(P) > 0$. This proves Assertion (3) for the invariant a_0.

Fix $1 \le i < d$. By Lemma 3.17.1, $P^{i/d} \in \Psi_{\frac{i}{d}}(M, V)$. Set

$$P_{i,\varepsilon} := P + \varepsilon P^{(d-i)/d} \quad \text{for} \quad \varepsilon > 0\,.$$

Since $(i - m)/d \notin \mathbb{N}$, $a_i(P_{i,\varepsilon})$ is given by a local formula and hence, since we are perturbing the lower order terms, is polynomial in ε. Thus either it vanishes identically or it is non-zero for all but a finite number of values of ε. We compute

$$\partial_\varepsilon \mathrm{Tr}_{L^2}\{e^{-tP_{i,\varepsilon}}\}|_{\varepsilon=0} = -t\mathrm{Tr}_{L^2}\{P^{(d-i)/d}e^{-tP}\}\,.$$

A straightforward computation with the Mellin transform along the lines developed in Section 1.3.4 yields

$$\begin{aligned}
& \partial_\varepsilon\{\Gamma(s)\zeta(s, P_{i,\varepsilon})\}|_{\varepsilon=0} \\
= & \int_0^\infty t^{s-1}\partial_\varepsilon \mathrm{Tr}_{L^2}\{e^{-tP_{i,\varepsilon}}\}|_{\varepsilon=0}dt \\
= & -\int_0^\infty t^{(s+1)-1}\mathrm{Tr}_{L^2}\{P^{(d-i)/d}e^{-tP}\}dt \\
= & -\Gamma(s+1)\zeta(s+1+\tfrac{i-d}{d}, P) = -\Gamma(s+1)\zeta(s+\tfrac{i}{d}, P)\,.
\end{aligned}$$

Since $1 \le i < d$, Γ is regular at $s = \frac{m-i}{d} + 1$. Consequently,

$$\begin{aligned}
\partial_\varepsilon a_i(P_{i,\varepsilon})|_{\varepsilon=0} &= -\Gamma(\tfrac{m-i}{d}+1)\mathrm{res}_{s=\frac{m-i}{d}}\zeta(s+\tfrac{i}{d}, P) \\
&= -\Gamma(\tfrac{m-i}{d}+1)\Gamma(\tfrac{m}{d})^{-1}\mathrm{res}_{s=\frac{m}{d}}\{\Gamma(s)\zeta(s, P)\} \\
&= -\Gamma(\tfrac{m-i}{d}+1)\Gamma(\tfrac{m}{d})^{-1}a_0(P) \ne 0\,.
\end{aligned}$$

It now follows that $a_i(\cdot)$ is generically non-zero; this establishes Assertion (3).

The proof of Assertion (4) is similar. Let $d = 2\bar{d}$ and set

$$P_{\vec{\varepsilon}} := D_2^{\bar{d}} + \varepsilon_{d-1}D_2^{\bar{d}-1} + ... + \varepsilon_0 \mathrm{Id} \in P_d(M, V)\,.$$

The argument given to establish Assertion (3) shows $a_n(P_{\vec{\varepsilon}})$ for $0 \le n < d$ and n even is non-zero for generic values of $\vec{\varepsilon}$ generically non-zero. Assertion (1) then yields $a_n(P_{\vec{\varepsilon}})$ generically non-zero for all even n. Fix such a value of $\vec{\varepsilon}$. Suppose that $R \in P_d(M, V)$. Let

$$R_\varrho := \varrho R + (1-\varrho)P_{\vec{\varepsilon}} \quad \text{for} \quad \varrho \in [0, 1]\,.$$

The invariants $a_n(R_\varrho)$ are analytic in ϱ. They do not vanish for $\varrho = 0$ by assumption. Thus these analytic invariants do not vanish for generic values of ϱ which establishes Assertion (4).

Next, we study the invariant b_1. Let $P \in \Psi_d(M,V)$. The arguments given above show there exist small constants ϱ_1 and ϱ_2 so that

$$a_{m+1}(Q) \neq 0 \quad \text{where} \quad Q := P^{2/d} + \varrho_1 P^{1/d} + \varrho_2 \in \Psi_2(M,V)\,.$$

Since Γ is regular at $s = -1/2$, Theorem 3.17.4 implies

$$\operatorname{res}_{s=-\frac{1}{2}}\{\zeta(s,Q)\} \neq 0\,. \tag{3.17.b}$$

We have $Q^{1/2} \in \Psi_1(M,V)$. Since $\zeta(s,Q^{1/2}) = \zeta(\frac{s}{2},Q)$, Equation (3.17.b) shows that $\zeta(s,Q^{1/2})$ has a non-trivial simple pole at $s = -1$. Therefore Theorem 3.17.4 shows

$$b_1(Q^{1/2}) = -\operatorname{res}_{s=-1}\{(s+1)\Gamma(s)\zeta(s,Q^{1/2})\} \neq 0\,.$$

We apply Assertion (2) to choose ϱ_3 small so

$$b_d(R) \neq 0 \quad \text{where} \quad R := Q^{1/2} + \varrho_3 \mathrm{Id} \in \Psi_1(M,V)\,.$$

We now have that $\Gamma(s)\zeta(s,R)$ has a non-trivial double pole at $s = -d$ so $\zeta(s,R)$ has a non-trivial residue at $s = -d$. Consequently

$$b_1(R^d) = \operatorname{res}_{s=-1}\{(s+1)\Gamma(s)\zeta(ds,R)\} \neq 0\,.$$

We note that R^d and P have the same leading symbol. We now set

$$R_{\varrho_4} := (1-\varrho_4)P + \varrho_4 R^d \in \Psi_d(M,V)\,.$$

The invariant $b_1(R_{\varrho_4})$ is analytic in ϱ_4 and non-trivial as $b_1(R^d) \neq 0$. Therefore $b_1(R_{\varrho_4}) \neq 0$ for all but a countable number of values of ϱ_4. Assertion (5) now follows.

To prove Assertion (6), we let $n = m + dk$ for $k > 0$. Suppose that $a_n(\cdot)$ is given by a local formula on $\Psi_d(M,V)$. Fix a Riemannian metric g on M and a Hermitian connection ∇ on V. Let $D_2 = D_2(g,\nabla)$ be defined by Equation (3.17.a) and set

$$\begin{aligned} P(g,\nabla;\vec{\varrho}): \;&=\; \varrho_4\{(D_2 + \varrho_1 D_2^{1/2} + \varrho_2 \mathrm{Id})^{1/2} + \varrho_3 \mathrm{Id}\}^d \\ &+\; (1-\varrho_4)D_2^{d/2} \in \Psi_d(M,V)\,. \end{aligned} \tag{3.17.c}$$

The invariants b_k are analytic in $\vec{\varrho}$ and, by the argument given to prove Assertion (5), non-trivial.

For $c > 0$, set $P_c(g,\nabla;\vec{\varrho}) := c^d P(g,\nabla;\vec{\varrho})$. We have

$$c^2 D_2(g;\nabla) = D_2(c^{-2}g;\nabla)\,.$$

Consequently, Equation (3.17.c) implies

$$P_c(g,\nabla;\vec{\varrho}) = P(c^{-2}g,\nabla;c\varrho_1,c^2\varrho_2,c\varrho_3,\varrho_4)\,.$$

Assume that a_n is given by a local formula. The analysis performed in Section 1.7 shows that the integrands comprising a_n would rescale according to a power law in c. On the other hand, it is immediate from the defining asymptotic series that

$$b_k(P_c(g,\nabla;\vec{\varrho})) = c^k b_k(P(g,\nabla;\vec{\varrho})), \quad \text{and}$$
$$a_n(P_c(g,\nabla;\vec{\varrho})) = c^k a_n(P(g,\nabla;\vec{\varrho})) + \log(c) c^k b_k(P(g,\nabla;\vec{\varrho}))\,.$$

Since the $\log(c)$ appears with a non-trivial coefficient, a_n is **not** given by a local formula. $\square$

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